A Preview of Monte Carlo Simulation Tools

The advent of new regulatory requirements, such as the Basel accord, and the dramatic growth of computing power have made Monte Carlo simulation an increasingly important and attractive technique with numerous applications in finance, especially in the areas of risk management and derivative pricing.

Furthermore, and not surprisingly, numerous financial clients have requested additional MATLAB tools to support simulation-based activities. To address these requests, The MathWorks has been actively developing a host of new and enhanced tools to directly support Monte Carlo simulation and related techniques.

This presentation will preview new and upcoming functionality specifically related to Monte Carlo simulation, such as:

- Probability distributions and related modeling tools, including extreme value theory (EVT) and piecewise distributions, as well as interactive tools for visualizing and fitting data
- Copula methods, including calibration and simulation of Gaussian & t copulas
- Stochastic differential equations, including stochastic interpolation and Brownian bridges
- Variance reduction techniques, including antithetic and stratified sampling

In addition to Monte Carlo, the demonstration also highlights a number of application areas of particular interest to financial clients, such as:

- Database connectivity & data importation
- Variety of optimization examples
- Graphical analysis & visualization capabilities
- Some new & advanced MATLAB language features (MCOS and function handles)
- Cell evaluation & publishing capabilities of MATLAB, which have been particularly popular with risk management professionals as a way documenting techniques & methodologies for colleagues & auditors alike

Contents

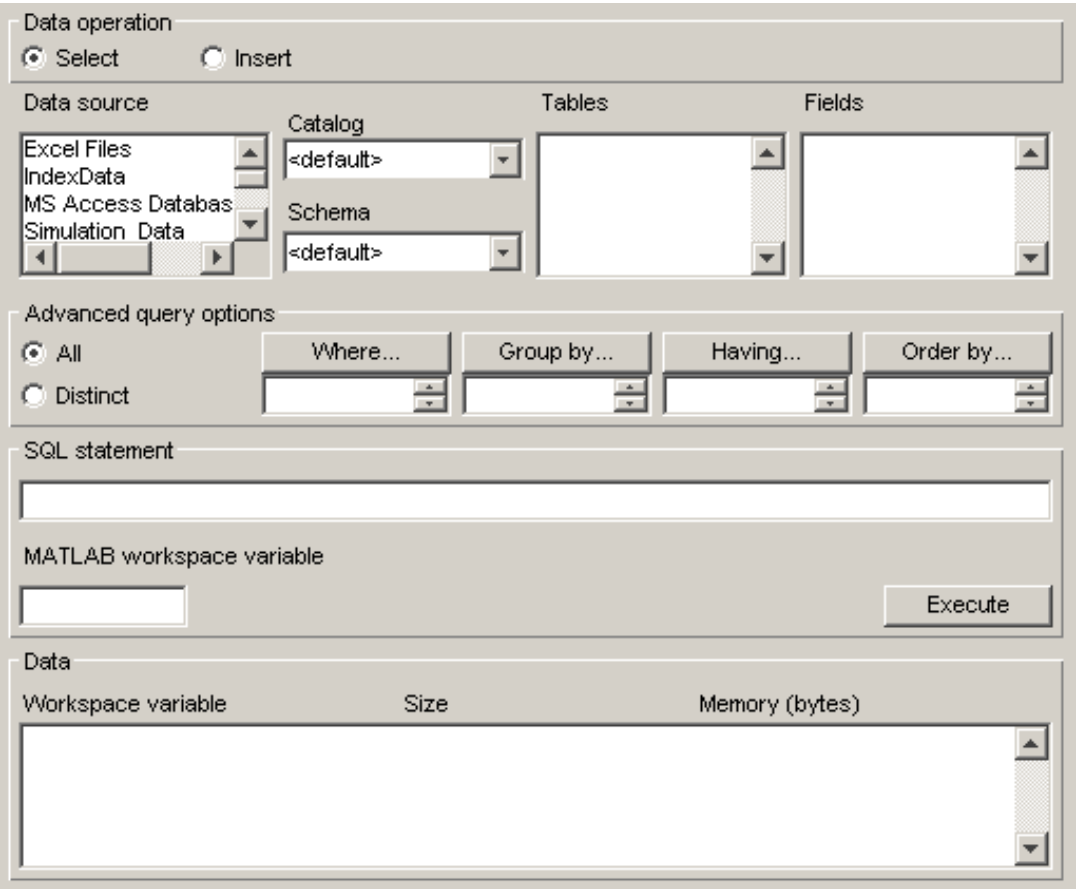
Import the Supporting Historical Dataset

To support the following presentation, first load a daily historical dataset of 3-month Euribor (quoted as an annual percentage rate, converted to daily effective yield), the closing index levels of representative large-cap equity indices of Canada (TSX Composite), France (CAC 40), Germany (DAX), Japan (Nikkei 225), UK (FTSE 100), and US (S&P 500), and corresponding trading dates spanning the interval 07-Feb-2001 to 24-Apr-2006.

There are numerous sources from which the supporting data may be imported into MATLAB, including various data service providers by way of the Datafeed Toolbox, spreadsheets applications, and directly from native binary MATLAB files (i.e., `MAT-files`).

The following code segment invokes the *Visual Query Builder* of the Database Toolbox, a graphical interface that allows users to import information from a database without knowing SQL.

```
clc, clear, querybuilder
```



The equivalent code needed to import the same dataset makes a database connection, then opens a cursor with the same SQL statement auto-generated by the Visual Query Builder, and finally fetches the historical data formatted as a data structure,

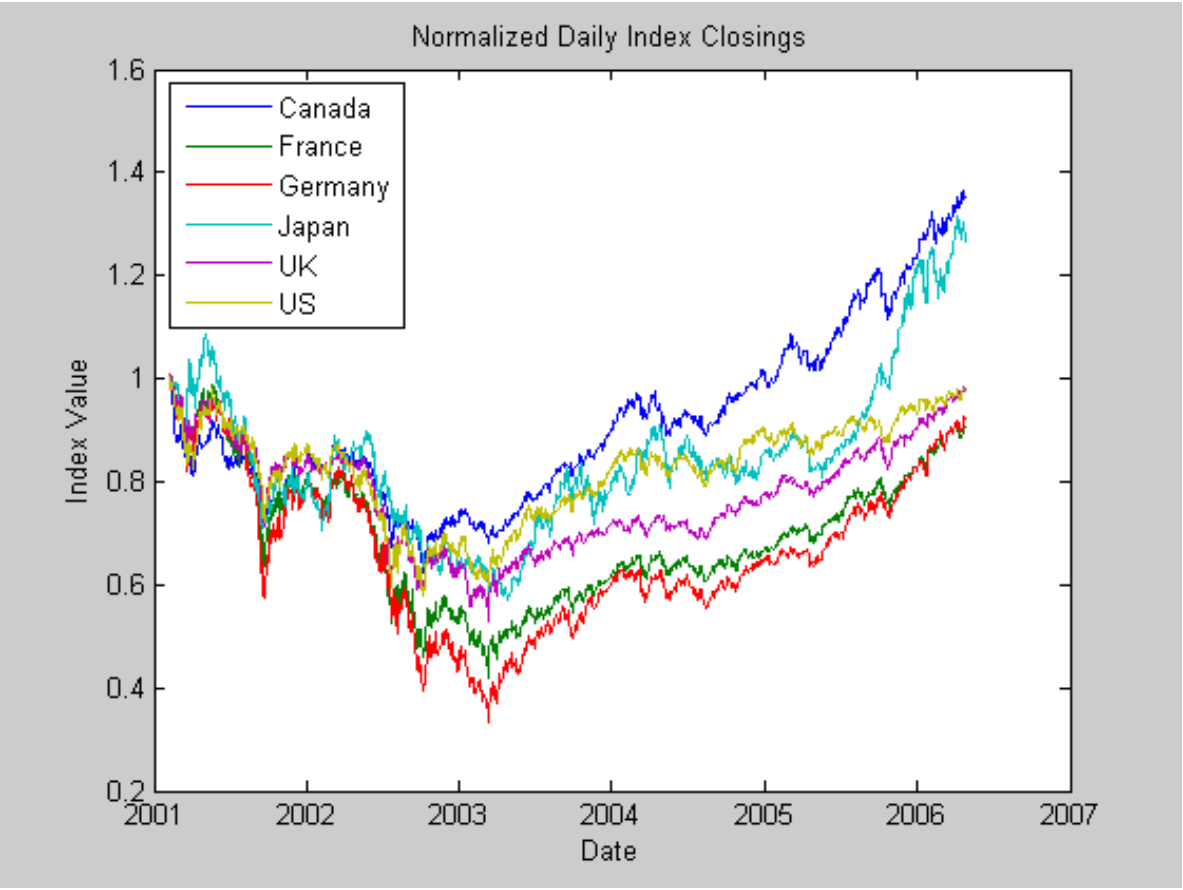
```
setdbprefs('DataReturnFormat', 'structure')
connection = database('Simulation_Data', '', '');
cursor     = exec(connection, 'SELECT ALL Dates,Canada,France,Germany,Japan,UK,US,
Euribor3M FROM Simulation_Data ORDER BY Dates ASC');
cursor     = fetch(cursor);
close(cursor), close(connection)

SDE_Data = cursor.Data
```

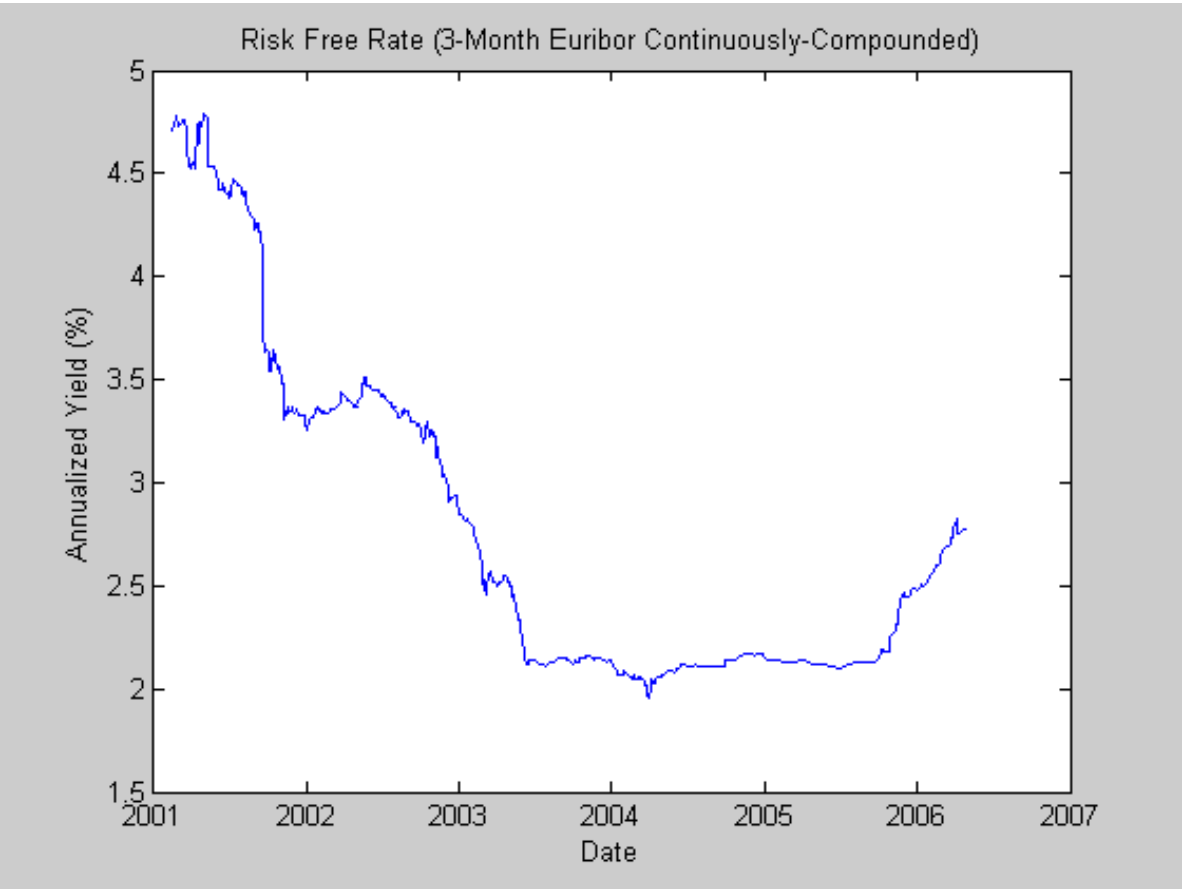
```
SDE_Data =
        Dates: [1359x1 double]
```

```
      Canada: [1359x1 double]
      France: [1359x1 double]
     Germany: [1359x1 double]
       Japan: [1359x1 double]
          UK: [1359x1 double]
          US: [1359x1 double]
   Euribor3M: [1359x1 double]
```

The following plots illustrate the data just imported. Specifically, we plot the relative price movements of each index as well as the Euribor risk-free rate proxy. Notice that the initial level of each index has been normalized to unity to facilitate the comparison of relative performance over the historical record.

```
fields    = fieldnames(SDE_Data);
dates     = SDE_Data.Dates;
countries = fields(2:end-1);
yields    = SDE_Data.Euribor3M;             % daily effective yields
yields    = 360 * log(1 + yields);          % continuous, annualized yields

for i = length(countries):-1:1
    prices(:,i) = SDE_Data.(countries{i});  % daily closing prices
end

figure, plot(dates, ret2price(price2ret(prices))), datetick('x')
xlabel('Date'), ylabel('Index Value'), title ('Normalized Daily Index Closings')
legend(countries, 'Location', 'NorthWest')

figure, plot(dates, 100 * yields)
datetick('x'), xlabel('Date'), ylabel('Annualized Yield (%)')
title('Risk Free Rate (3-Month Euribor Continuously-Compounded)')
```





Extreme Value Theory, Piecewise Distributions, and Interactive Fitting Tools

In recent years, numerous financial clients have requested additional functionality related to the use of *Extreme Value Theory*, a statistical tool for modeling the fat-tailed behavior of financial data such asset returns and insurance losses.

In response, 2 univariate probability distributions have been added to the Statistics Toolbox:

- Generalized Extreme Value (GEV) distribution, which lends itself to a modeling technique known as the *block maxima or minima* method. In this approach, an historical dataset is divided into a set of sub-intervals, or blocks, and the largest or smallest observation in each block is recorded and fitted to a GEV distribution.
- Generalized Pareto (GP) distribution, which lends itself to a modeling technique known as the *distribution of exceedances* or *peaks over threshold* method. In this approach, an historical dataset is sorted, and the amount by which those observations which exceed a specified threshold is fitted to a GP distribution.

Since the latter approach is more popular in risk management applications, the following analysis highlights the Pareto distribution.

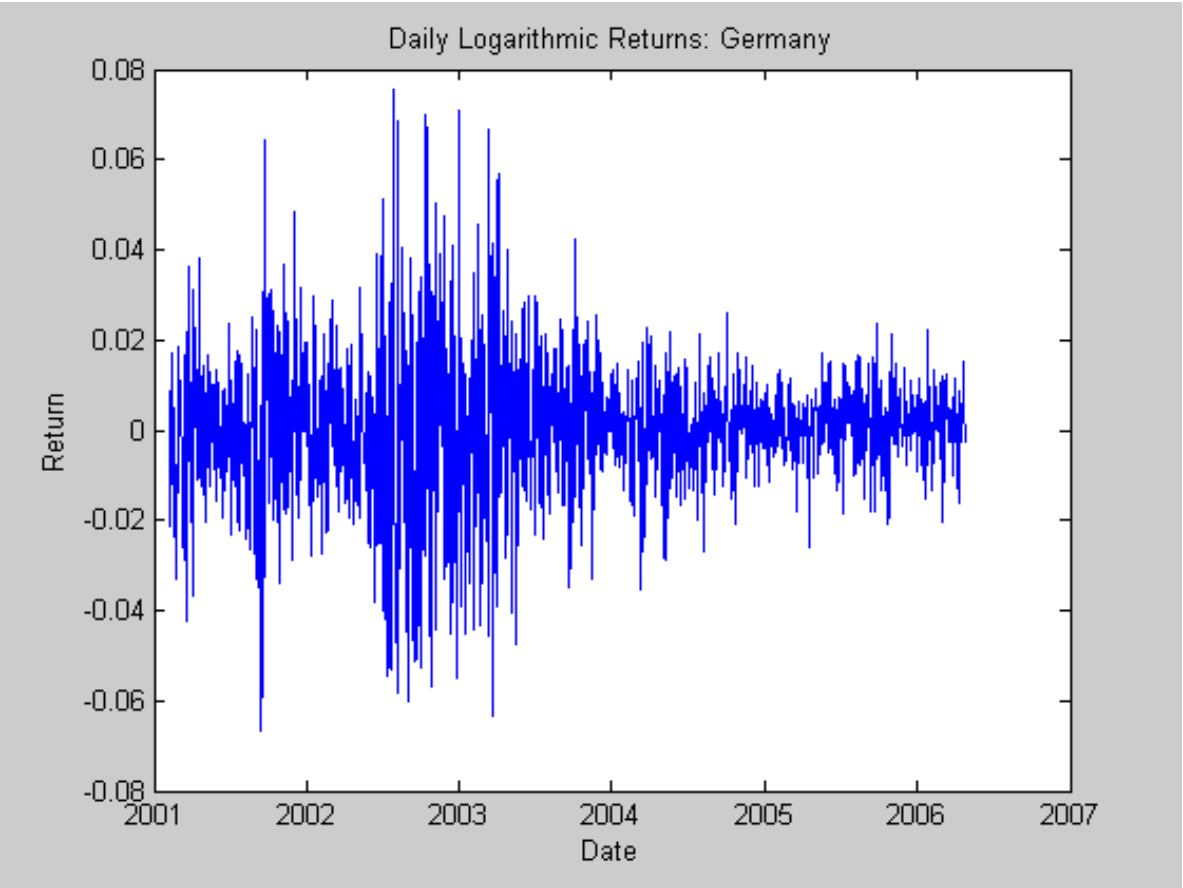
In particular, suppose we wish to provide a complete statistical description of the probability distribution of daily asset returns of any one of the equity indices. Furthermore, assume that this description is provided by a piecewise semi-parametric distribution in which the asymptotic behavior in each tail is characterized by a generalized Pareto distribution.

First convert the closing level of each equity index to daily logarithmic returns (sometimes called geometric, or continuously compounded, returns).

```
returns       = price2ret(prices);  % logarithmic returns
[T, nIndices] = size(returns);      % historical sample size and # of indices
```

Since the following analysis applies extreme value theory to characterize the distribution of each individual equity index return series, it is helpful to examine the details for a particular country. The next line of code can be changed to examine the details for any country.

```
index = strmatch('Germany', countries); if isempty(index), index = 1; end

figure, plot(dates(2:end), returns(:,index)), datetick('x')
xlabel('Date'), ylabel('Return')
title(['Daily Logarithmic Returns: ' countries{index}])
```



Given the daily returns shown above, estimate the empirical, or non-parametric, CDF of each index with a Gaussian kernel. This smoothes the CDF estimates, eliminating the staircase pattern of unsmoothed sample CDFs. Although non-parametric kernel CDF estimates are well suited for the interior of the distribution where most of the data is found, they tend to perform poorly when applied to the upper and lower tails. To better estimate the tails of the distribution, apply EVT to those residuals that fall in each tail.

Specifically, find upper and lower thresholds such that 10% of the residuals is reserved for each tail. Then fit the amount by which those extreme residuals in each tail fall beyond the associated threshold to a parametric GP distribution by maximum likelihood.

The following code segment creates objects of type **paretotails**, one such object for each index return series. These Pareto tail objects encapsulate the estimates of the parametric GP lower tail, the non-parametric kernel-smoothed interior, and the parametric GP upper tail to construct a composite semi-parametric CDF for each index.

The resulting piecewise distribution object allows interpolation within the interior of the CDF and extrapolation (function evaluation) in each tail. Extrapolation is very desirable, allowing estimation of quantiles outside the historical record, and is invaluable for risk management applications.

Moreover, Pareto tail objects also provide methods to evaluate the CDF and inverse CDF (quantile function), and to query the cumulative probabilities and quantiles of the boundaries between each segment of the piecewise distribution.

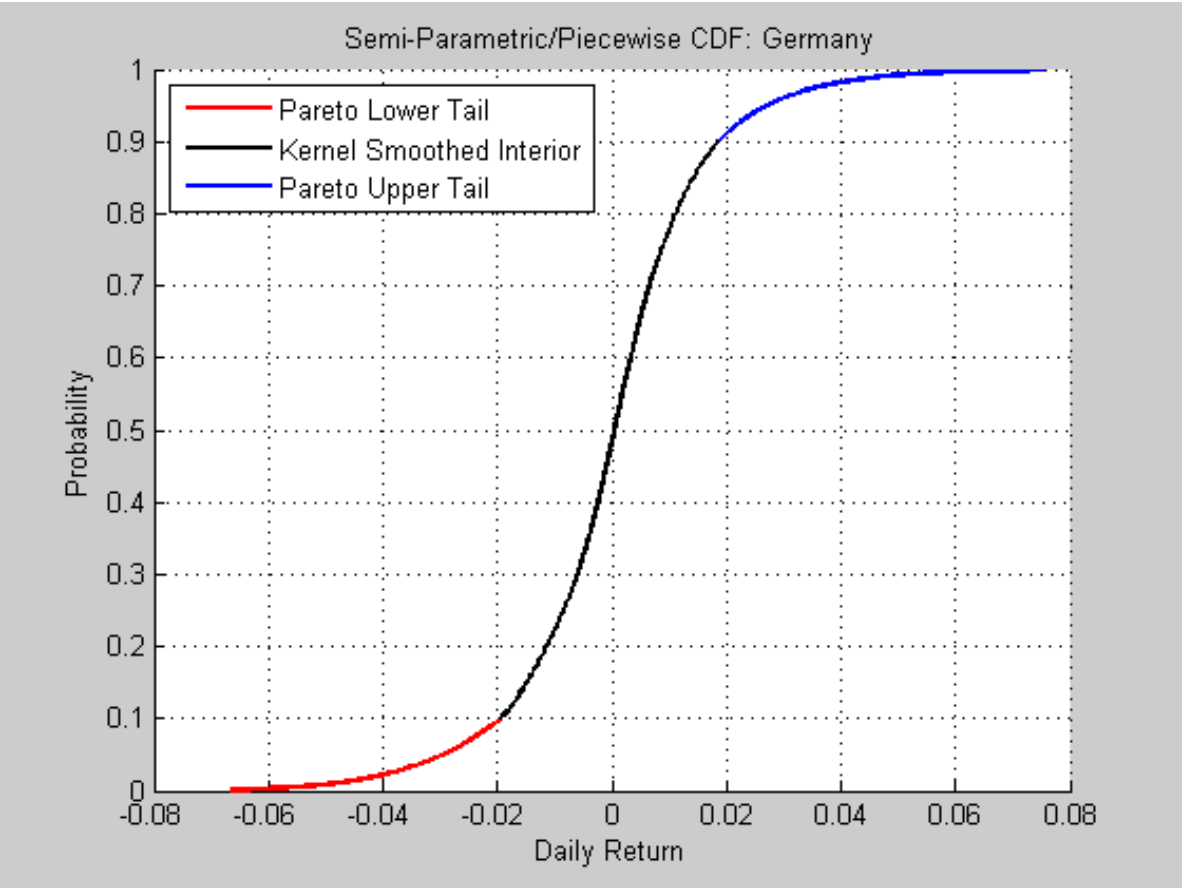
Also, notice that collections of Pareto tail objects are stored in cell arrays, high-level MATLAB data containers designed to store disparate data types.

```
nPoints      = 200;      % # of sampled points of kernel-smoothed CDF
tailFraction = 0.1;      % Decimal fraction of residuals allocated to each tail

OBJ = cell(nIndices,1);  % Cell array of Pareto tail objects

for i = 1:nIndices
    OBJ{i} = paretotails(returns(:,i), tailFraction, 1 - tailFraction, 'kernel');
end
```

Having estimated the three distinct regions of the piecewise distribution, graphically concatenate and display the result. Again, note that the lower and upper tail regions, displayed in red and blue, respectively, are suitable for extrapolation, while the kernel-smoothed interior, in black, is suitable for interpolation.

The code below calls the CDF and inverse CDF methods of the Pareto tails object of interest with data other than that upon which the fit is based. Specifically, the referenced methods have access to the fitted state, and are now invoked to select and analyze specific regions of the probability curve, acting as a powerful data filtering mechanism.

```
figure, hold('on'), grid('on')

minProbability = OBJ{index}.cdf((min(returns(:,index))));
maxProbability = OBJ{index}.cdf((max(returns(:,index))));

pLowerTail = linspace(minProbability  , tailFraction    , 200); % sample lower tail
pUpperTail = linspace(1 - tailFraction, maxProbability  , 200); % sample upper tail
pInterior  = linspace(tailFraction    , 1 - tailFraction, 200); % sample interior

plot(OBJ{index}.icdf(pLowerTail), pLowerTail, 'red'  , 'LineWidth', 2)
plot(OBJ{index}.icdf(pInterior) , pInterior , 'black', 'LineWidth', 2)
plot(OBJ{index}.icdf(pUpperTail), pUpperTail, 'blue' , 'LineWidth', 2)

xlabel('Daily Return'), ylabel('Probability')
title (['Semi-Parametric/Piecewise CDF: ' countries{index}])
legend({'Pareto Lower Tail' 'Kernel Smoothed Interior' ...
        'Pareto Upper Tail'}, 'Location', 'NorthWest')
```



Although the previous graph illustrates the composite CDF, it is instructive to examine the GP fit in more detail.

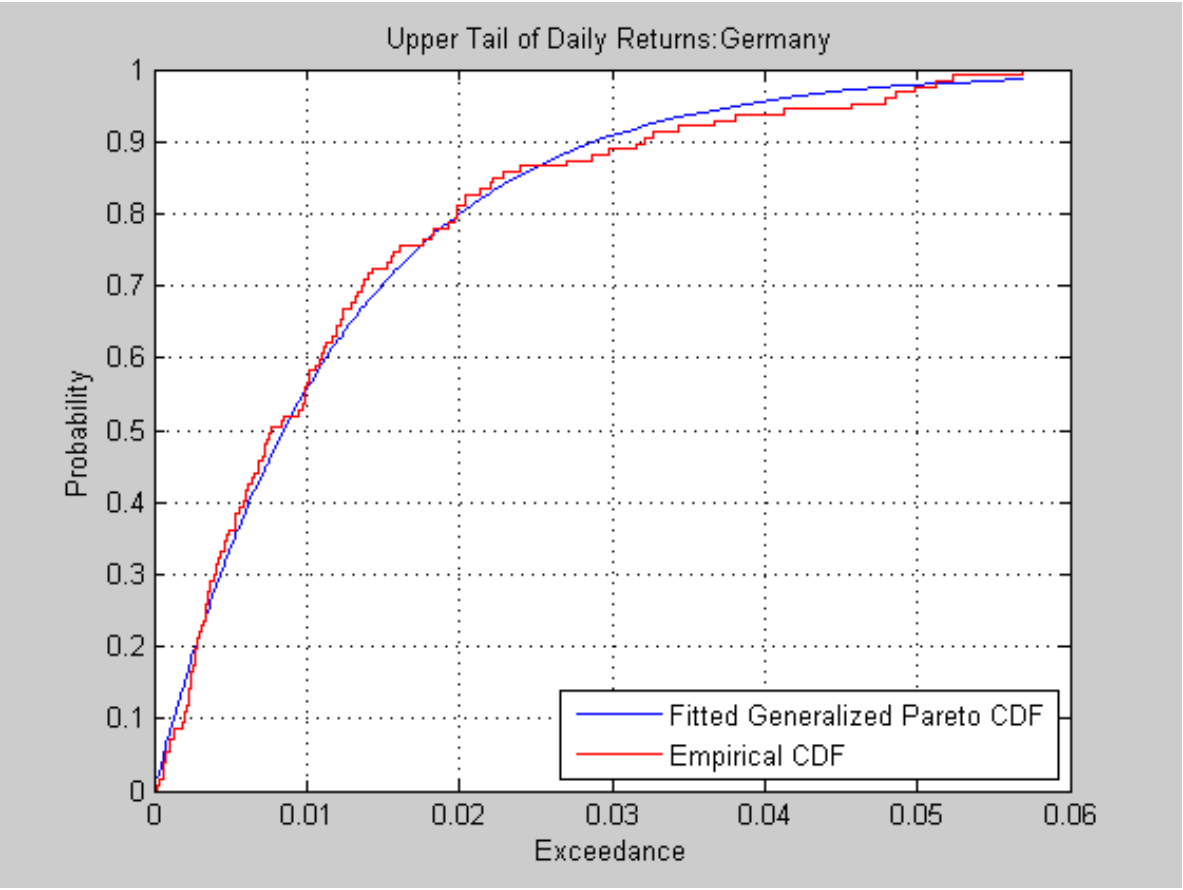
The CDF of a GP distribution is parameterized as

$$F(y) = 1 - (1 + \zeta y/\beta)^{-1/\zeta}, y >= 0, \beta > 0, \zeta > -0.5$$

for exceedances (y), tail index parameter (zeta), and scale parameter (beta).

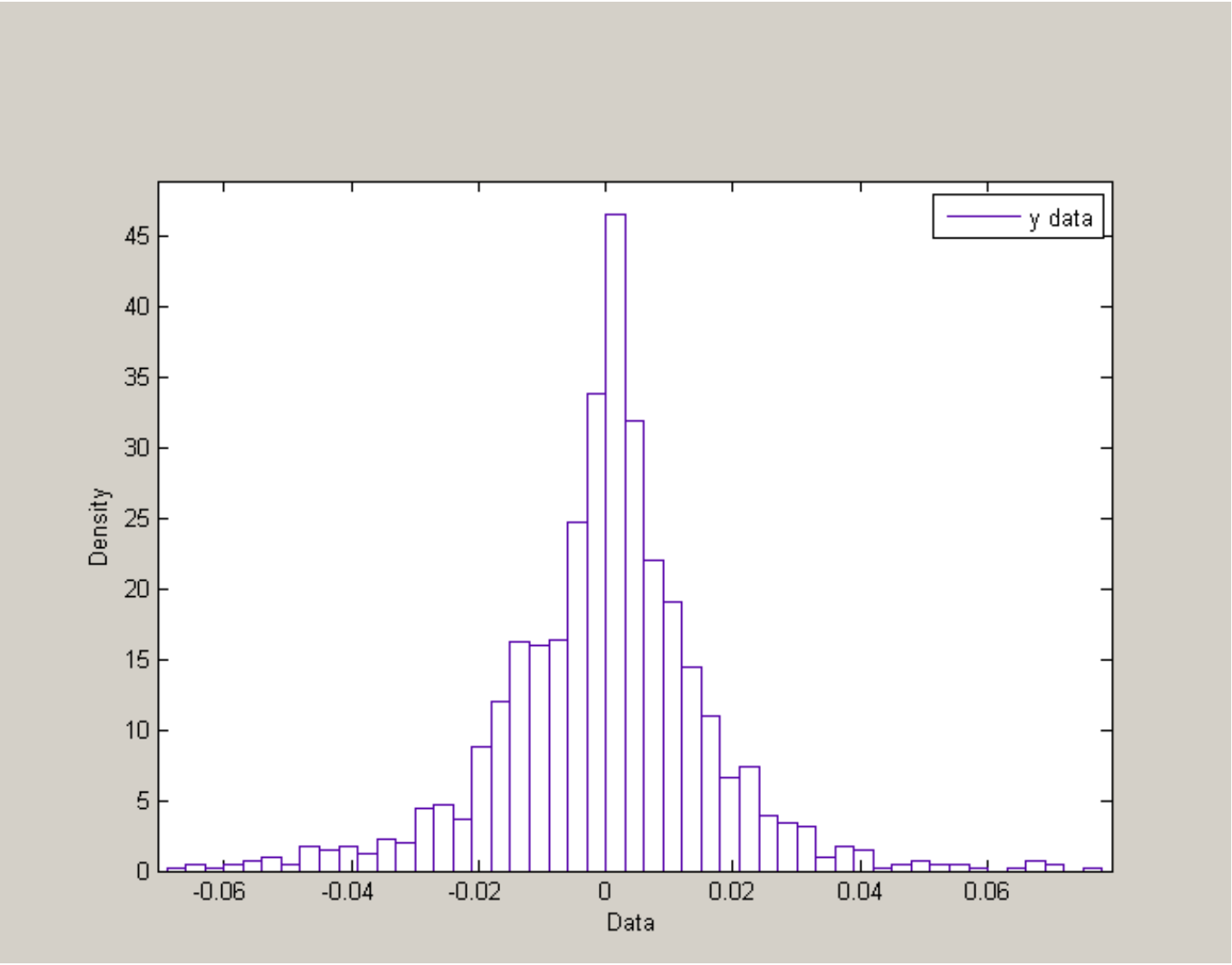
To visually assess the GP fit, plot the empirical CDF of the upper tail exceedances of the returns along with the CDF fitted by the GP distribution. Although only 10% of the standardized residuals is used, the fitted distribution closely follows the exceedance data, so the GP model seems to be a good choice.

```
figure
[P,Q] = OBJ{index}.boundary;     % cumulative probabilities & quantiles at boundaries
y     = sort(returns(returns(:,index) > Q(2), index) - Q(2)); % sort exceedances
plot(y, (OBJ{index}.cdf(y + Q(2)) - P(2))/P(1))
[F,x] = ecdf(y);                 % empirical CDF
hold('on'); stairs(x, F, 'r'); grid('on')

legend('Fitted Generalized Pareto CDF','Empirical CDF','Location','SouthEast');
xlabel('Exceedance'); ylabel('Probability');
title (['Upper Tail of Daily Returns:' countries{index}])
```



Although the analysis shown immediately above is suitable for inclusion within a production environment, it may also be performed interactively by invoking the *Distribution Fitting Tool* found in the Statistics Toolbox.

```
dfittool(returns(:,index))
```



Copula Methods for Gaussian & Student's t Copulas

Recently, the Statistics Toolbox has added new functionality specifically related to the calibration and simulation of Gaussian and t copulas.

A copula is a multi-variate cumulative distribution function (CDF) with uniformly-distributed margins. Although the theoretical foundations were established decades ago, copulas have experienced a tremendous surge in popularity over the last few years, primarily as a technique for modeling non-Gaussian portfolio risks.

Although numerous families exist, all copulas represent a statistical device for modeling the dependence structure between 2 or more random variables. In addition, important statistics, such as *rank correlation* and *tail dependence*, are properties of a given copula and are unchanged by monotonic transforms of its margins.

Since the CDF and inverse CDF (i.e., quantile function) of the piecewise distributions described above are both monotonic transforms, a copula provides a very convenient technique to simulate dependent random variables with dissimilar and arbitrarily-distributed margins. Moreover, since a copula defines a given dependence structure irrespective of its margins, calibration is typically much easier than estimating the joint distribution function.

Copula Calibration

Given the daily index returns from above, now estimate the parameters of the Gaussian and t copulas using the new Statistic Toolbox function **copulafit**. Since a t copula becomes a Gaussian copula as the scalar degrees of freedom parameter (DoF) becomes infinitely large, the two copulas are really of the same family, and therefore share a linear correlation matrix as a fundamental parameter.

Although calibration of the linear correlation matrix of a Gaussian copula is straightforward, the calibration of a t copula is more difficult. For this reason, the Statistics Toolbox offers 2 techniques to calibrate a t copula.

The first technique performs maximum likelihood estimation (MLE) in a two-step process. The inner step maximizes the log-likelihood with respect to the linear correlation matrix, given a fixed value for the degrees of freedom. That conditional maximization is placed within a 1-D maximization with respect to the degrees of freedom, thus maximizing the log-likelihood over all parameters. The function being maximized in this outer step is known as the profile log-likelihood for the degrees of freedom.

The second technique is derived by differentiating the log-likelihood function with respect to the linear correlation matrix, assuming the degrees of freedom is a fixed constant. The resulting expression is a non-linear equation that can be solved iteratively for the correlation matrix. This technique approximates the profile log-likelihood for the degrees of freedom parameter for large sample sizes. This technique is usually significantly faster than the true maximum likelihood technique outlined above, however, it should be used with caution because the estimates and confidence limits may not be accurate for small or moderate sample sizes.

The code segment below first transforms the daily returns to uniform variates by the piecewise, semi-parametric CDFs derived above, then fits the Gaussian and t copulas to the transformed data. When the uniform variates are transformed by the empirical CDF of each margin, the calibration method is often known as canonical maximum likelihood (CML).

```
U = zeros(size(returns));

for i = 1:nIndices
    U(:,i) = OBJ{i}.cdf(returns(:,i));    % transform each margin to uniform
end

options     = statset('Display', 'off', 'TolX', 1e-4);
[rhoT, DoF] = copulafit('t', U, 'Method', 'ApproximateML', 'Options', options);
rhoG        = copulafit('Gaussian', U); clc
```

If we examine the estimated correlation matrices, we see that they are quite similar but not identical.

```
rhoG     % linear correlation matrix of the optimized Gaussian copula
```

```
rhoG =
    1.0000    0.4745    0.5018    0.1857    0.4721    0.6622
    0.4745    1.0000    0.8606    0.2393    0.8459    0.4912
    0.5018    0.8606    1.0000    0.2126    0.7608    0.5811
    0.1857    0.2393    0.2126    1.0000    0.2396    0.1494
    0.4721    0.8459    0.7608    0.2396    1.0000    0.4518
    0.6622    0.4912    0.5811    0.1494    0.4518    1.0000
```

```
rhoT     % linear correlation matrix of the optimized t copula
```

```
rhoT =
    1.0000    0.4662    0.4843    0.1912    0.4735    0.6512
    0.4662    1.0000    0.8878    0.2574    0.8502    0.5119
    0.4843    0.8878    1.0000    0.2337    0.7730    0.5866
    0.1912    0.2574    0.2337    1.0000    0.2510    0.1542
    0.4735    0.8502    0.7730    0.2510    1.0000    0.4771
    0.6512    0.5119    0.5866    0.1542    0.4771    1.0000
```

Moreover, notice the relatively low degrees of freedom parameter obtained from the t copula calibration, indicating a significant departure from a Gaussian situation.

```
DoF      % scalar degrees of freedom parameter of the optimized t copula
```

```
DoF =
    4.8612
```

Copula Simulation

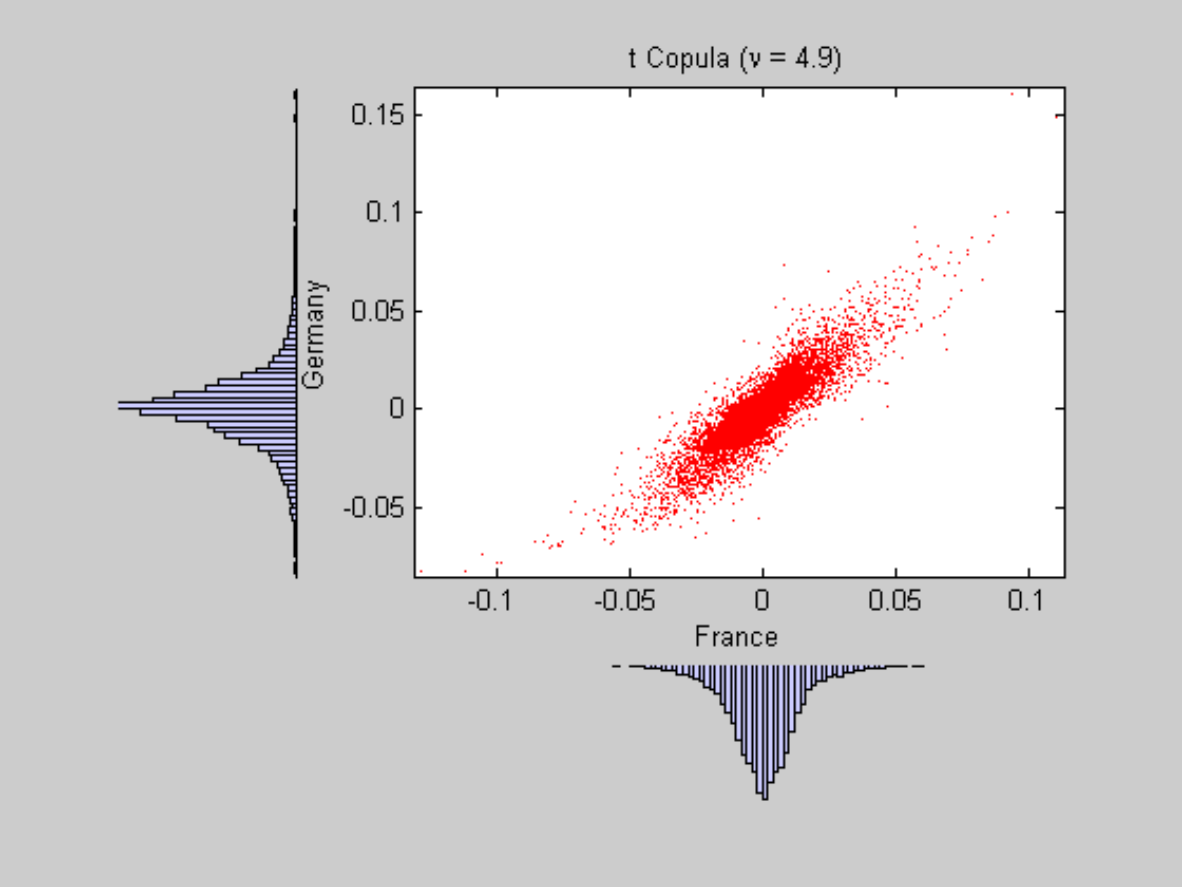
Now that the copula parameters have been estimated, simulate jointly-dependent equity index returns using the **copularnd** function of the Statistics Toolbox.

Then, by extrapolating into the GP tails and interpolating into the smoothed interior, transform the uniform variates derived from **copularnd** to daily returns via the inverse CDF of each index. This produces simulated returns consistent with those obtained from the historical dataset. These returns are assumed independent in time, but at any point in time possess the dependence and rank correlation induced by the given copula.

The following code segment simulates index returns using the t copula, and plots a 2-D scatter plot with marginal histograms for the French CAC 40 and German DAX using the new **scatterhist** function of the Statistics Toolbox (the French and German indices were chosen simply because they have the highest correlation of the available data).

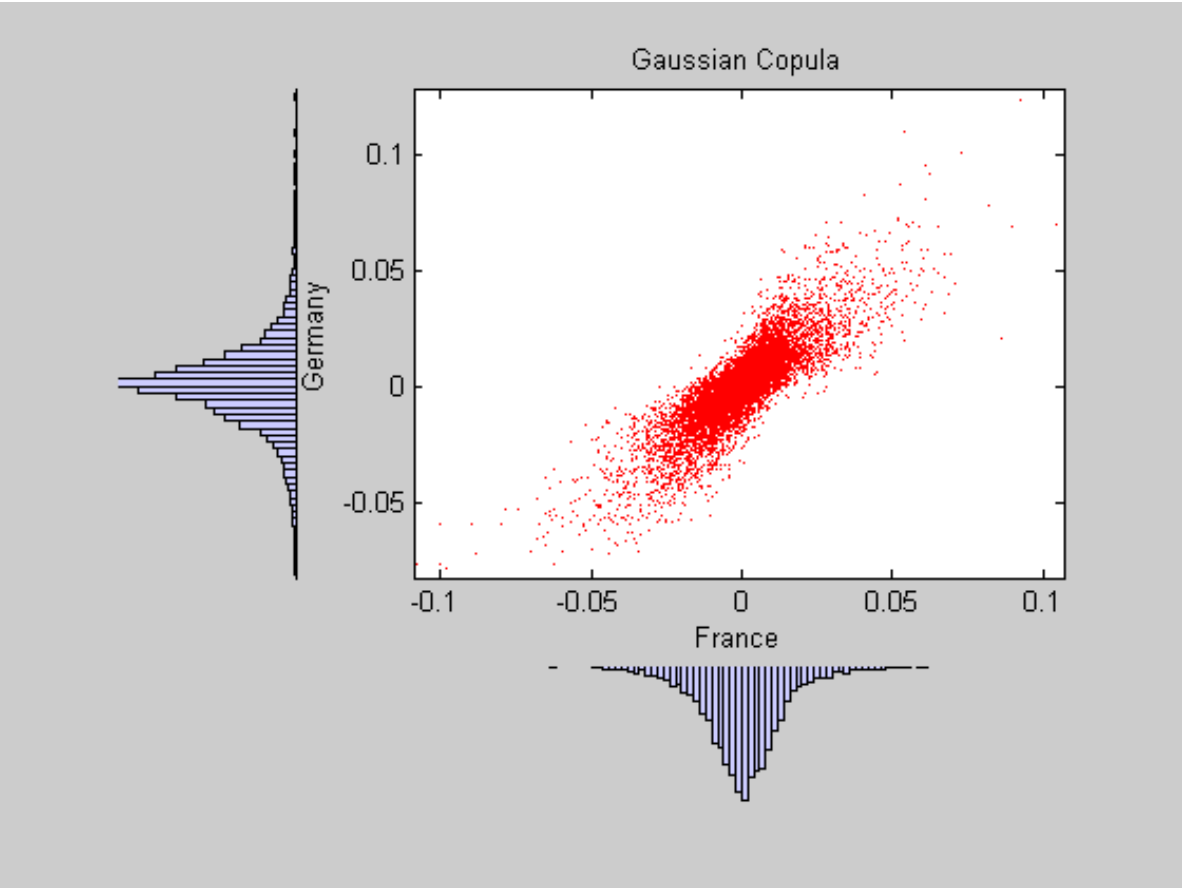
```
horizon = 10000;                          % # of observations (simulation horizon)
randn('state', 0), rand('twister', 0)

R = zeros(horizon, nIndices);             % pre-allocate simulated returns array
U = copularnd('t', rhoT, DoF, horizon);   % simulate U(0,1) from t copula

for j = 1:nIndices
    R(:,j) = OBJ{j}.icdf(U(:,j));
end

figure
h = scatterhist(R(:,2), R(:,3));
set(findobj(h(1), 'Type', 'line'), 'marker', '.', 'color', 'red', 'markerSize', 1)
```

```
xlabel('France'), ylabel('Germany'), title(['t Copula (\nu = ' num2str(DoF,2) ')'])
```



Now simulate and plot index returns using the Gaussian copula.

```
randn('state', 0), rand('twister', 0)

R = zeros(horizon, nIndices);              % pre-allocate simulated returns array
U = copularnd('Gaussian', rhoG, horizon); % simulate U(0,1) from Gaussian copula

for j = 1:nIndices
    R(:,j) = OBJ{j}.icdf(U(:,j));
end

figure
h = scatterhist(R(:,2), R(:,3));
set(findobj(h(1), 'Type', 'line'), 'marker', '.', 'color', 'red', 'markerSize', 1)
xlabel('France'), ylabel('Germany'), title('Gaussian Copula')
```



To conclude the discussion of copulas, examine the 2 figures just created. In particular, notice the remarkable similarity between the miniature histograms on the corresponding axes of each figure. This similarity is no coincidence.

Since both copulas simulate uniform random variables, which are then transformed to daily returns by the inverse CDF of the piecewise distribution of each index, the simulated returns of any given index are identically distributed regardless of the copula. However, the scatter graph of each figure indicates the dependence structure associated with the copula, and in contrast to the uni-variate margins shown in the histograms, the scatter graphs are quite distinct.

Once again, the copula defines a dependence structure irrespective of its margins, and therefore offers many appealing and convenient features not limited to calibration alone.

Monte Carlo Simulation of Stochastic Differential Equations

The following sections highlight important new features and functionality specifically designed to allow financial clients to simulate Stochastic Differential Equations (SDEs) in MATLAB.

Interestingly, most clients that participated in MATLAB advisory board/focus group discussions indicated that they were primarily interested in efficient simulation methods, and were largely content to calibrate their own models, citing the complexity associated with model calibration as *an art rather than a science*.

The architecture is completely new and makes extensive use of some of the latest MATLAB language features, including object oriented programming capabilities built upon the new MATLAB Common Object System (MCOS) infrastructure and data encapsulation using nested functions.

High Level Features

Although many of the following features are discussed below and demonstrated by way of a series of examples, it useful to highlight some of the more salient features here:

- Euler approximation default simulation method
- Approximate analytic solution for separable geometric Brownian motion and Hull-White/Vasicek models
- Specialized methods for efficient simulation of static, separable Geometric Brownian Motion and Brownian Motion multivariate models
- Vectorized methods for efficient simulation of static univariate models
- Stochastic interpolation & Brownian bridge simulation methods
- Full support for any combination of static and dynamic model parameters
- Full support for state and Brownian vectors of arbitrary dimensionality
- Optional user-specified random number generation & dependence/correlation structure
- Antithetic sampling
- End-of-period processing/state vector adjustments to perform virtually any type of path-dependent analysis
- Ability to sample an SDE at intermediate times without reporting those times, improving accuracy and reducing memory burden
- Ability to avoid storing state and noise time series to improve performance and memory efficiency

General Parametric Specification

The proposed SDE engine allows the simulation of generalized multivariate stochastic processes, and is designed to provide a flexible and powerful simulation architecture. In addition, the framework provides several utilities and model classes, offering users a variety of parametric specifications and interfaces.

The architecture is fully multi-dimensional in the both the state vector as well as the Brownian motion, and offers users the convenience of linear and mean-reverting drift rate specifications. Most parameters may be specified as traditional MATLAB arrays or as functions accessible by a common interface, thereby supporting rather general dynamic/non-linear relationships common in SDE simulations.

Specifically, the architecture allows users to simulate correlated paths of any number of state variables driven by a vector-valued Brownian motion of arbitrary dimensionality, thereby approximating the underlying multivariate continuous-time process by a vector-valued stochastic difference equation.

Consider the following general stochastic differential equation,

$$dX_t = F(t, X_t)dt + G(t, X_t)dW_t$$

where X is an `NVARS` x 1 state vector of process variables (e.g., short rates, equity prices) we wish to simulate, dW is an `NBROWNS` x 1 Brownian motion vector (also referred to as a Wiener process), F is an `NVARS` x 1 vector-valued drift rate function, and G is an `NVARS` x `NBROWNS` matrix-valued diffusion rate function.

Notice that the drift and diffusion rates, F and G, respectively, are rather general functions of a real-valued scalar sample time t and state vector X(t). Also, notice that static (non-time-varying) coefficients are simply a special case of the more general dynamic (time-varying) situation, in the same manner as a function may be a trivial constant, e.g., f(t,X) = 4.

The above SDE is quite general, and so it is often useful to implement derived classes which impose additional structure on the drift and diffusion rate functions.

Consider the following linear drift rate specification,

$$F(t, X_t) = A(t, X_t) + B(t, X_t)X_t$$

where A is an NVARS x 1 vector-valued function and B is an NVARS x NVARS matrix-valued function.

As an alternative, consider a drift rate specification expressed in mean-reverting form,

$$F(t, X_t) = S(t, X_t)[L(t, X_t) - X_t]$$

S is an NVARS x NVARS matrix-valued function of mean reversion speeds (i.e., rates of mean reversion), and L is an NVARS x 1 vector-valued function of mean reversion levels (i.e., long run average level).

Similarly, consider the following diffusion rate specification,

$$G(t, X_t) = D(t, X_t^{\alpha(t, X_t)})V(t, X_t)$$

where D is an NVARS x NVARS diagonal matrix-valued function in which each diagonal element is the corresponding element of the state vector raised to the corresponding element of an exponent Alpha, which is also an NVARS x 1 vector-valued function. V is an NVARS x NBROWNS matrix-valued function of instantaneous volatility rates in which each row corresponds to a particular state variable and each column to a particular Brownian source of uncertainty, and associates the exposure of state variables with sources of risk.

The parametric specifications for the drift and diffusion rate functions are designed to impose a sense of structure, more intuitively associating parametric restrictions with familiar models derived from the general SDE class.

Common Interface and Behavior

As already mentioned, much of the design is driven by a desired look and feel. In fact, most models and utilities outlined below are MATLAB objects, and share common behavior obtained by intentionally blurring the lines between methods, user-defined functions, object properties, and fields of traditional data structures. Therefore, in what follows, properties and methods are collectively referred to as *members* in an attempt to be intentionally vague.

Although the following examples elaborate in more detail, it is important to highlight the fact that dynamic (i.e., time-varying) behavior is associated with function evaluation. Moreover, this dynamic behavior is accessible by passing time and state (t,X) to a common, published interface, and is pervasive throughout the SDE class system. Although this (t,X) interface appears limited, this seemingly simple function evaluation approach may be used to model or construct powerful analytics, especially when used with concert with nested functions.

Throughout many of the examples that follow, most model members may be evaluated, or invoked, as would any MATLAB function. Thus, although it may be helpful to examine and access object members in a manner similar to that of data structures, it is often more convenient and useful to think of them as functions that perform an action.

Example: Incorporating Dynamic Behavior

As previously mentioned, object members are designed to be evaluated as if they were MATLAB

functions accessible by a common interface. This accessibility, in turn, gives users the impression of dynamic behavior regardless of whether or not the underlying parameters are truly time-varying. Since members are accessible by a common interface, seemingly simple, linear constructs may in fact represent complex, non-linear designs.

Moreover, when members are entered as functions object constructors may only verify that they return arrays of correct size by evaluating them at the initial time and state, but otherwise have no knowledge of any particular functional form. This is both flexible and potentially dangerous.

Most users will create objects by entering traditional MATLAB arrays that explicitly represent constant (non-dynamic) parameters, or will need to convert arrays that represent historical time series.

Since time series arrays represent dynamic behavior, and dynamic behavior must be captured by functions accessible by the (t,X) interface, users will need utilities to convert traditional time series arrays into callable functions of time and state. To address this, a special conversion function called **ts2func** (i.e., time series to function) is available.

To illustrate dynamic behavior, the following example works with the daily historical dataset of 3-month Euribor rates and closing levels of the French CAC 40. Assume we wish to simulate risk-neutral sample paths of the CAC 40 index using a geometric Brownian motion (GBM) model,

$$dX_t = r_t X_t dt + \sigma X_t dW_t$$

in which

$$r_t$$

represents evolution of the risk-free rate of return.

Furthermore, assume we wish to annualize the relevant information derived from the daily data, and that each calendar year is composed of 250 trading days.

```
clc
dt      = 1 / 250;                      % time increment = 1 day = 1/250 years
returns = price2ret(SDE_Data.France);   % daily log returns of CAC 40
sigma   = std(returns) * sqrt(250);     % annualized volatility

yields  = SDE_Data.Euribor3M;
yields  = 360 * log(1 + yields);        % continuously-compounded, annual yield
```

Now suppose we wish to compare the resulting sample paths obtained from two risk-neutral historical simulation approaches in which the daily Euribor yields serve as a proxy for the risk-free rate of return.

The first approach specifies the risk-neutral return as the sample average of Euribor yields, and therefore assumes a constant (non-dynamic) risk-free return.

```
nPeriods = length(yields);              % # of simulated observations

randn('state', 25)
obj    = gbm(mean(yields), diag(sigma), 'StartState', 100)
[X1,T] = obj.simulate(nPeriods, 'DeltaTime', dt);
```

```
obj =
   Class GBM: Generalized Geometric Brownian Motion
   -------------------------------------------------
     Dimensions: State = 1, Brownian = 1
   -------------------------------------------------
      StartTime: 0
     StartState: 100
    Correlation: 1
          Drift: drift rate function F(t,X(t))
      Diffusion: diffusion rate function G(t,X(t))
     Simulation: simulation method/function simByEuler
         Return: 0.0278117
          Sigma: 0.231875
```

In contrast, the second approach specifies the risk-neutral return as the historical time series of Euribor yields, and therefore assumes a dynamic, yet deterministic, rate of return (i.e., this example does ***not*** illustrate stochastic interest rates). To illustrate this dynamic effect, use the **ts2func** utility,

```
r = ts2func(yields, 'Times', (0:nPeriods - 1)'); clc
```

Notice that the **ts2func** utility packages a specified time series array inside a callable function of time and state, and also synchronizes it with an optional time vector. For instance,

```
r(0,100)
```

```
ans =
    0.0470
```

evaluates the function at (t = 0, X = 100) and returns the first observed Euribor yield. However, notice that the resulting function may also be evaluated at any intermediate time t and state X,

```
r(7.5,200)
```

```
ans =
    0.0472
```

Furthermore, notice that the following command produces exactly the same result when called with time alone,

```
r(7.5)
```

```
ans =
    0.0472
```

The equivalence of these last two commands highlights some important features of the architecture.

When members are specified as functions, they must evaluate properly when passed a scalar, real-valued sample time (t) followed by a state vector (X), and generate an array of appropriate dimensions (which in the first case is a scalar constant, and in the second is a scalar, piecewise constant function of time alone).

Notice that there is absolutely no obligation to use either time (t) or state (X). In fact, the caller has no knowledge of any implementation details. In this respect the architecture is only publishing an interface: it specifies what the inputs and outputs must be, but otherwise offers users complete flexibility regarding implementation.

In the current example, the function does indeed evaluate properly when passed time followed by state, thereby satisfying the minimal requirements. The fact that it also evaluates correctly when passed only time simply indicates that the function does not require the state vector X. The important point to make is that it works when passed (t,X)!

Furthermore, notice that the utility **ts2func** performs a zero-order-hold (ZOH) piecewise constant interpolation, and that the notion of piecewise constant parameters is pervasive throughout the SDE architecture.

To complete the comparison, perform the second simulation using the same initial random number state,

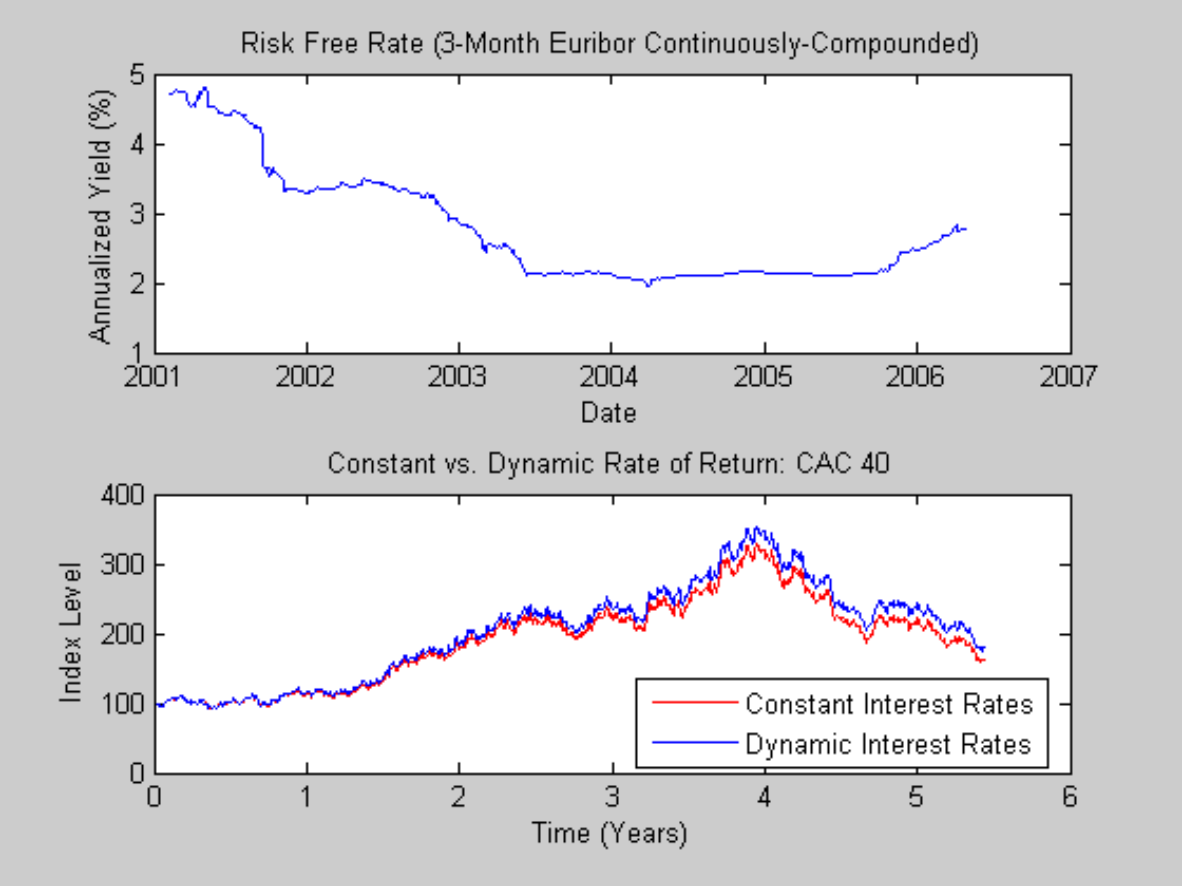
```
randn('state', 25), clc
obj = gbm(r, diag(sigma), 'StartState', 100)
X2  = obj.simulate(nPeriods, 'DeltaTime', dt);
```

```
obj =
   Class GBM: Generalized Geometric Brownian Motion
   -------------------------------------------------
     Dimensions: State = 1, Brownian = 1
   -------------------------------------------------
      StartTime: 0
     StartState: 100
    Correlation: 1
          Drift: drift rate function F(t,X(t))
      Diffusion: diffusion rate function G(t,X(t))
     Simulation: simulation method/function simByEuler
         Return: function ts2func/vector2Function
          Sigma: 0.231875
```

Finally, plot the series of risk-free reference rates and compare the two simulation trials,

```
figure, subplot(2,1,1)
plot(SDE_Data.Dates, 100 * yields)
datetick('x'), xlabel('Date'), ylabel('Annualized Yield (%)')
title('Risk Free Rate (3-Month Euribor Continuously-Compounded)')

subplot(2,1,2)
plot(T, X1, 'red', T, X2, 'blue')
```

```
xlabel('Time (Years)'), ylabel('Index Level')
title('Constant vs. Dynamic Rate of Return: CAC 40')
legend({'Constant Interest Rates'  'Dynamic Interest Rates'}, 'Location', 'Best')
```



Notice that the paths are quite close, but not identical. The blue line in the bottom plot uses all the historical Euribor data, and illustrates a single trial of an historical simulation.

Example: The Brownian Bridge Part I: Stochastic Interpolation without Refinement

Many applications involve post-processing of simulated paths, and may require knowledge of the state vector at intermediate sample times initially unavailable. One way to approximate these intermediate states is to perform some sort of deterministic interpolation. However, deterministic interpolation techniques fail to capture the correct probability distribution at these intermediate times.

A better technique that captures the correct joint distribution performs a Brownian, or stochastic, interpolation by sampling from a conditional Gaussian distribution. This sampling technique is sometimes referred to as a *Brownian bridge*.

The default stochastic interpolation technique is designed to interpolate into an existing time series, ignoring any new interpolated states as additional information becomes available. This is the usual notion of interpolation, and is referred to as *interpolation without refinement*.

Alternatively, the interpolation technique may insert new interpolated states into the existing time series upon which subsequent interpolation is based, thereby refining the information available at subsequent interpolation times. This is referred to as *interpolation with refinement*.

Interpolation without refinement is a more traditional interpolation technique, and is most suitable when

the input series is closely spaced in time. In this situation, interpolation without refinement is a valuable tool for data imputation in the presence of missing information, but is generally unsuitable as a data extrapolation technique.

Interpolation with refinement is often more suitable when the input series is widely spaced, and is a useful tool for extrapolation. In particular, when extrapolation is performed, the interpolation method is in fact a valuable Monte Carlo simulation method in its own right, and nicely complements the suite of simulation methods.

Although the stochastic interpolation method is available to any model, the behavior is best illustrated by way of a constant-parameter Brownian motion process. Consider a correlated, bi-variate Brownian motion (BM) model of the form

$$dX_{1t} = 0.3dt + 0.2dW_{1t} - 0.1dW_{2t}$$

$$dX_{2t} = 0.4dt + 0.1dW_{1t} - 0.2dW_{2t}$$

$$E[dW_{1t}dW_{2t}] = \rho dt = 0.5dt$$

Create a BM object to represent the bi-variate model,

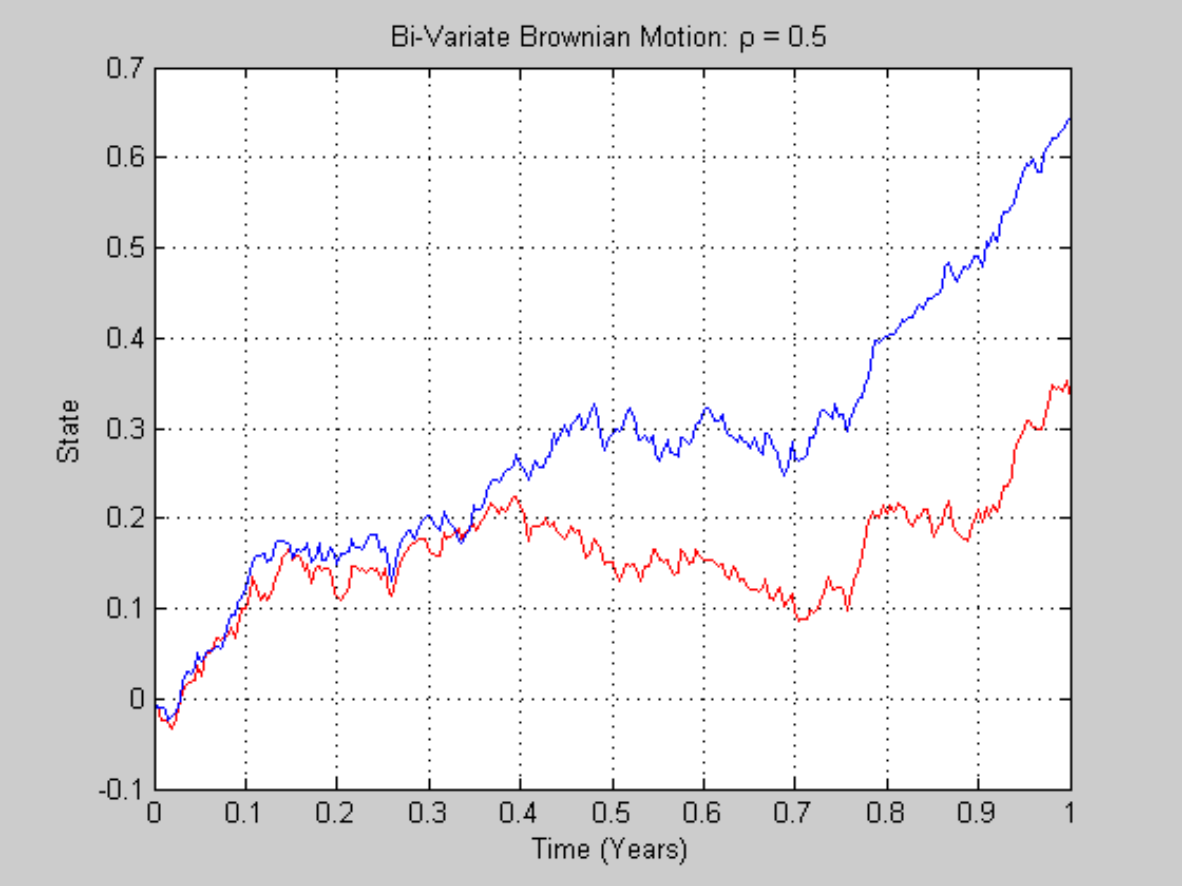
```
clc
mu    = [0.3 ; 0.4];
sigma = [0.2  -0.1 ; 0.1  -0.2];
rho   = [ 1    0.5 ; 0.5    1 ];

obj   = bm(mu,  sigma, 'Correlation', rho)
```

```
obj =
   Class BM: Brownian Motion
   -----------------------------------------
     Dimensions: State = 2, Brownian = 2
   -----------------------------------------
      StartTime: 0
     StartState: 0 (2x1 double array)
    Correlation: 2x2 double array
          Drift: drift rate function F(t,X(t))
      Diffusion: diffusion rate function G(t,X(t))
     Simulation: simulation method/function simByEuler
             Mu: 2x1 double array
          Sigma: 2x2 diagonal double array
```

Assume the drift (Mu) and diffusion (Sigma) parameters above are annualized, and simulate a single Monte Carlo trial of daily observations for one calendar year (i.e., 250 trading days).

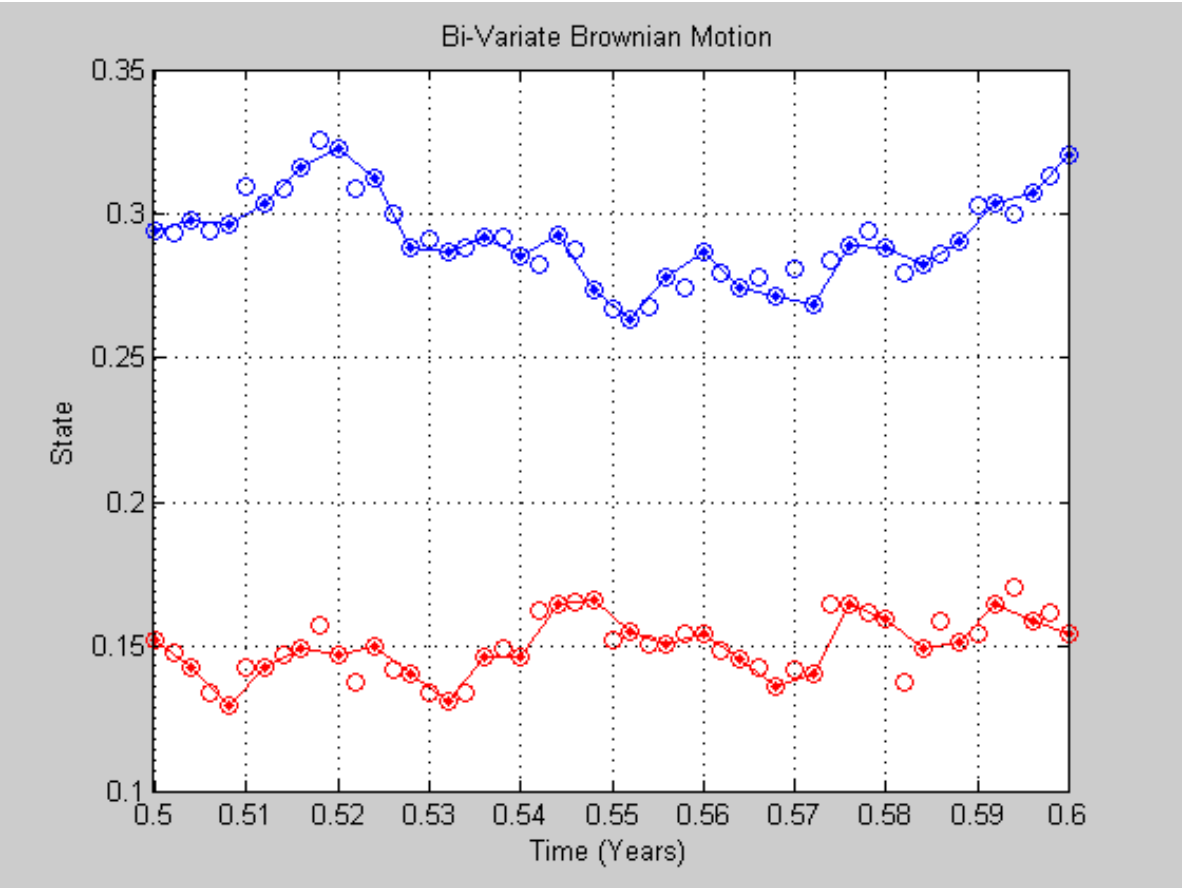
```
randn('state', 0)
dt    = 1/250;          % 1 trading day = 1/250 years
[X,T] = obj.simulate(250, 'DeltaTime', dt);
```

```
figure, plot(T, X(:,1), 'red', T, X(:,2), 'blue'), grid('on')
xlabel('Time (Years)'), ylabel('State')
title('Bi-Variate Brownian Motion: \rho = 0.5')
limits = axis; axis([0 1 limits(3:4)])
```



Now interpolate into the simulated time series with a Brownian bridge and plot both the simulated and interpolated values. To illustrate the effect of interpolation without refinement, it is helpful to examine a small interval in detail,

```
t = ((T(1) + dt/2) : (dt/2) : (T(end) - dt/2));
x = obj.interpolate(t, X, 'Times', T);

plot(T, X(:,1), '.-red', T, X(:,2), '.-blue'), grid('on'), hold('on')
plot(t, x(:,1), 'o red', t, x(:,2), 'o blue'), hold('off')

xlabel('Time (Years)'), ylabel('State')
title('Bi-Variate Brownian Motion')
axis([0.4999 0.6001 0.1 0.35])
```



In this plot, solid red and blue dots indicate the simulated states of the bi-variate model, and the straight lines connecting these solid dots indicate intermediate states that would be obtained from a deterministic linear interpolation. Open circles indicate interpolated states.

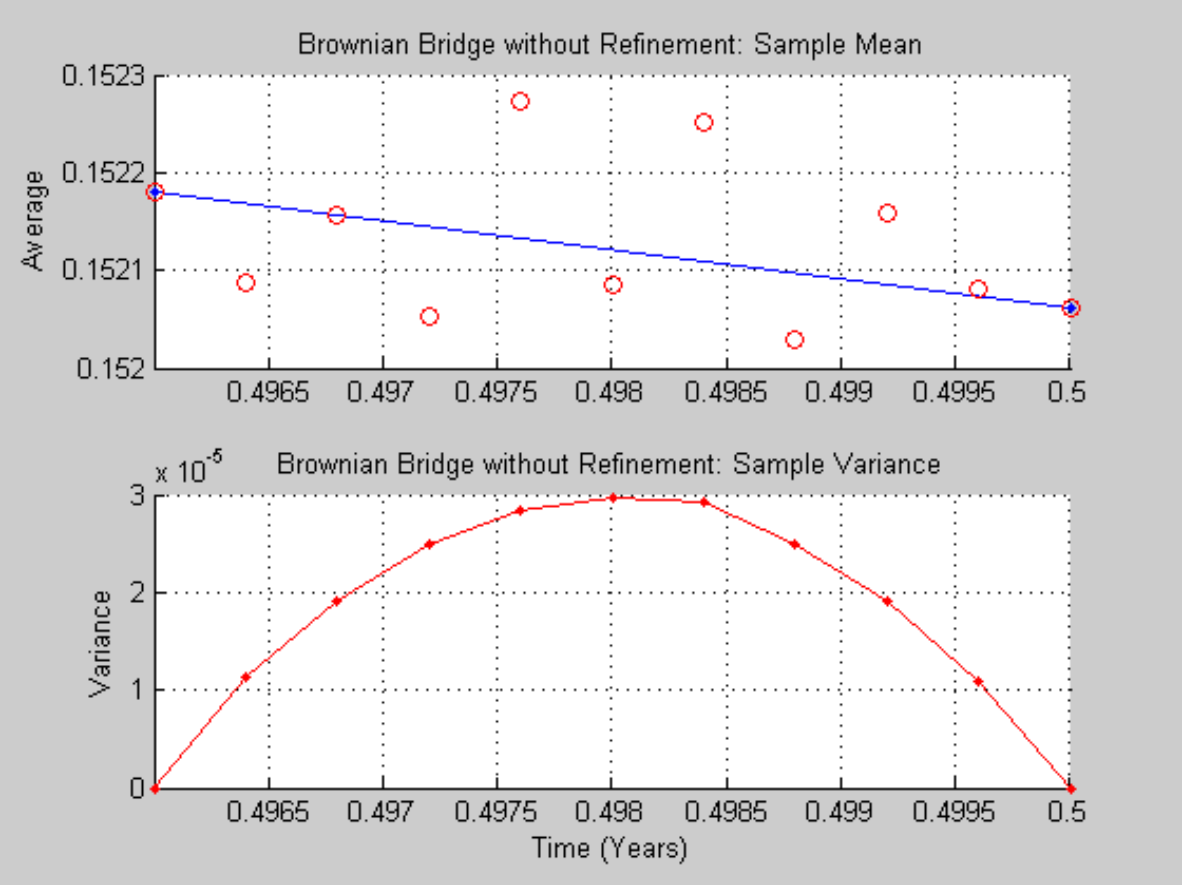
Notice that open circles associated with every other interpolated state encircle solid dots associated with the corresponding simulated state. However, interpolated states at the mid-point of each time increment typically deviate from the straight line connecting each solid dot.

Additional insight into the behavior of a stochastic interpolation is revealed by regarding a Brownian bridge as a Monte Carlo simulation of a conditional Gaussian distribution.

The following code segment examines the behavior of a Brownian bridge over a single time increment. Specifically, a single time increment of length `dt` is divided into 10 sub-intervals. In each sub-interval, independent draws are taken from a Gaussian distribution conditioned on the simulated states to the left and right, and the sample mean and variance of each is plotted. For clarity, the graph below plots the sample statistics of the first state variable only, but similar results hold for any state variable.

```
randn('state', 10)
nTrials  = 5000;           % # of Trials at each interpolation time
n        = 125;            % index of some simulated state near the middle

times    = (T(n) : (dt/10) : T(n + 1));
average  = zeros(length(times),1);
variance = zeros(length(times),1);

for i = 1:length(times)
    t = times(i);
```

```
    x = obj.interpolate(t(ones(nTrials,1)), X, 'Times', T);
    average(i)  = mean(x(:,1));
    variance(i) = var(x(:,1));
end

subplot(2,1,1), hold('on'), grid('on')
plot([T(n) T(n + 1)], [X(n,1) X(n + 1,1)], '.-b')
plot(times, average, 'or'), hold('off')
title('Brownian Bridge without Refinement: Sample Mean')
ylabel('Average')
limits = axis; axis([T(n) T(n + 1) limits(3:4)])

subplot(2,1,2), hold('on'), grid('on')
plot(T(n), 0, '.-b', T(n + 1), 0, '.-b')
plot(times, variance, '.-r'), hold('off')
title('Brownian Bridge without Refinement: Sample Variance')
xlabel('Time (Years)'), ylabel('Variance')
limits = axis; axis([T(n) T(n + 1) limits(3:4)])
```



The Brownian interpolation within the chosen `dt` interval illustrates the following:

- The conditional mean of each state variable lies on a straight line segment between the original simulated states at each endpoint.
- The conditional variance of each state variable is a quadratic function which attains its maximum midway between the interval endpoints, and is zero at each endpoint.
- The maximum variance, although dependent upon the actual model diffusion rate function G(t,X), is the variance of the sum of correlated Gaussian variates scaled by the factor dt/4.

The illustration above highlights interpolation without refinement, in that none of the interpolated states take into account any new information as it becomes available.

Had interpolation with refinement been performed, new interpolated states would have been inserted into the time series and made available to subsequent interpolations on a trial-by-trial basis. In this situation, all random draws for any given interpolation time would be identical. Had this been the case, the above plot of the sample mean would exhibit greater variability but would still be clustered around the straight line segment between the original simulated states at each endpoint. The plot of the sample variance, however, would be zero for all interpolation times, exhibiting no variability at all.

Example: The Brownian Bridge Part II: Stochastic Interpolation with Refinement

The previous example introduced the default stochastic interpolation technique known as *interpolation without refinement*. Without refinement, interpolated states are unavailable at subsequent interpolation times, making the technique generally unsuitable for data extrapolation.

Alternatively, an interpolation technique may insert new interpolated states into an existing time series upon which subsequent interpolation is based, making any new state information available at subsequent interpolation times.

This is referred to as *interpolation with refinement*, and is more suitable for data extrapolation. With refinement, the interpolation technique offers a legitimate Monte Carlo simulation method in its own right, and complements the suite of simulation methods.

For reference, notice that all simulation methods require users to specify a time grid by specifying the number of periods, optionally augmented with a scalar or vector of strictly positive time increments, and a number of intermediate time steps. The combination of these, together with an initial sample time associated with the object, uniquely determines the sequence of times at which the state vector is sampled. Thus, simulation methods traverse the time grid from beginning to end (i.e., from left to right).

In contrast, interpolation methods allow the time grid to be traversed in any order, allowing both forward and backward movements in time. To do so, they allow users to specify a vector of interpolation times, the elements of which are not required to be unique.

Since samples may be generated in any order, a traditional Monte Carlo simulation is just a special case of interpolation: an open-ended interpolation that involves sampling from a Gaussian distribution conditioned on the previous state alone.

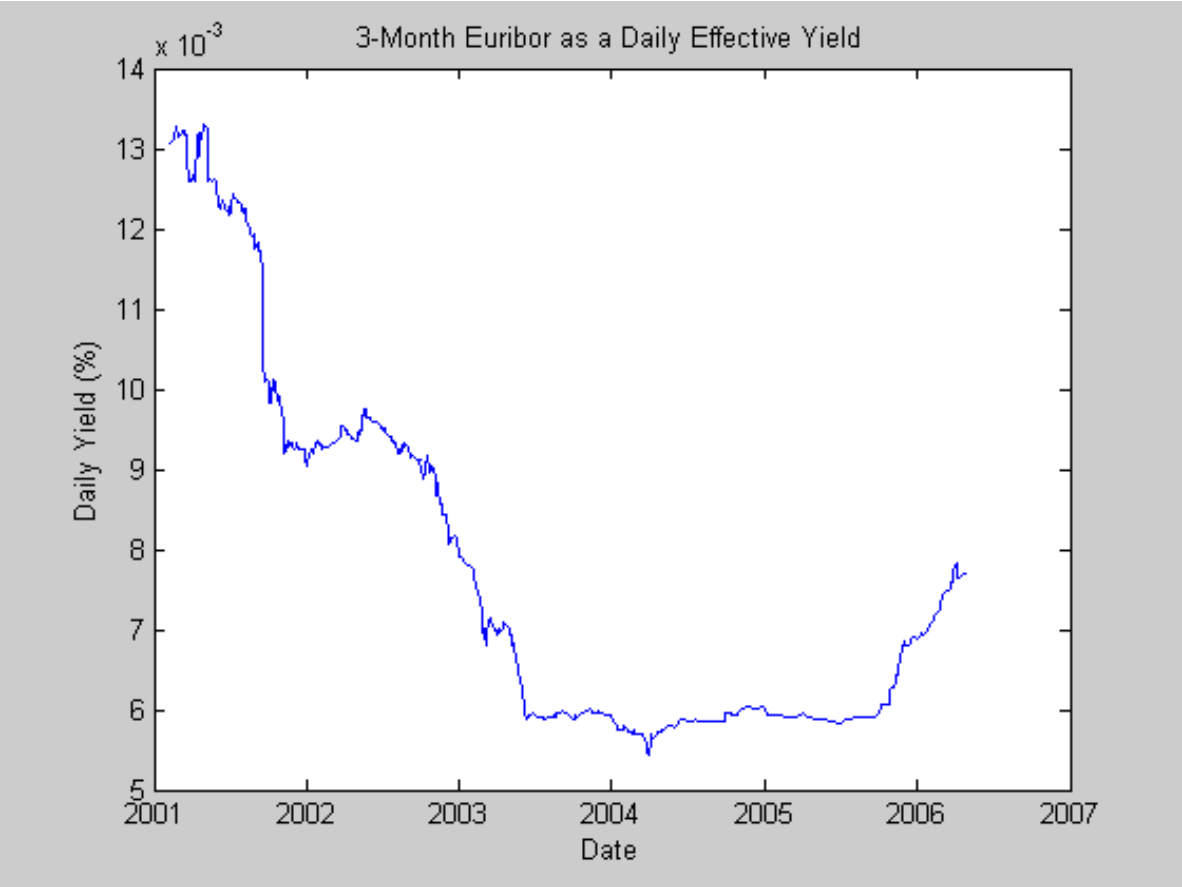
Note that many references define *The Brownian Bridge* as a conditional simulation combined with a scheme for traversing the time grid, effectively merging 2 distinct algorithms. In contrast, the interpolation method offered here provides additional flexibility by intentionally separating the algorithms.

One popular algorithm for moving about a time grid involves an initial Monte Carlo simulation to sample the state at the terminal time, then successively sampling intermediate states by stochastic interpolation. This algorithm allows the first few samples to determine the overall behavior of the paths, while later samples progressively refine the structure. Such algorithms are often cited as *variance reduction techniques*.

This algorithm is particularly simple when the number of interpolation times is a power of 2. In this case, each interpolation falls midway between 2 known states, refining the interpolation in a manner similar to bi-section. The following example highlights the flexibility of refined interpolation by implementing this popular power-of-two algorithm.

To illustrate this approach, we again work with the daily series of 3-month Euribor rates.

```
clf, plot(SDE_Data.Dates, 100 * SDE_Data.Euribor3M)
datetick('x'), xlabel('Date'), ylabel('Daily Yield (%)')
title('3-Month Euribor as a Daily Effective Yield')
```



Now consider fitting a simple univariate Vasicek model to the daily equivalent yields of the 3-month Euribor data,

$$dX_t = S(t, X_t)[L(t, X_t) - X_t]dt + V(t, X_t)dW_t$$

in which

$$S(t, X_t) = S,\ L(t, X_t) = L,\ V(t, X_t) = \sigma$$

are scalar constants. Given initial conditions, the distribution of the short rate at some time T in the future is Gaussian with mean

$$E(X_T) = X_0 e^{-ST} + L(1 - e^{-ST})$$

and variance

$$Var(X_T) = \sigma^2(1 - e^{-2ST})/2S$$

To calibrate this simple short rate model, re-write the model in more familiar regression format,

$$y_t = \alpha + \beta x_t + \epsilon_t$$

in which

$$y_t = dX_t,\ \alpha = SLdt,\ \beta = -Sdt,$$

and perform an ordinary linear regression in which the model volatility is proportional to the standard error of the residuals,

$$\sigma = \sqrt{Var(\epsilon_t)/dt}$$

```
yields     = SDE_Data.Euribor3M;
regressors = [ones(length(yields) - 1, 1)   yields(1:end-1)];

[coefficients, intervals, residuals] = regress(diff(yields), regressors);

dt    = 1;               % time increment = 1 day
speed = -coefficients(2) / dt;
level = -coefficients(1) / coefficients(2);
sigma =  std(residuals)  / sqrt(dt); clc
```

Now create a HWV object with an initial StartState set to the most recently observed short rate,

```
obj = hwv(speed, level, sigma, 'StartState', yields(end))
```

```
obj =
   Class HWV: Hull-White/Vasicek
   ---------------------------------------
     Dimensions: State = 1, Brownian = 1
   ---------------------------------------
      StartTime: 0
     StartState: 7.70408e-005
    Correlation: 1
          Drift: drift rate function F(t,X(t))
      Diffusion: diffusion rate function G(t,X(t))
     Simulation: simulation method/function simByEuler
          Sigma: 4.77637e-007
          Level: 6.00424e-005
          Speed: 0.00228854
```

Assume, for example, we wish to simulate the fitted model over 64 trading days (a power of 2), but using a refined Brownian bridge with the power-of-two algorithm instead of the usual beginning-to-end Monte Carlo simulation approach.

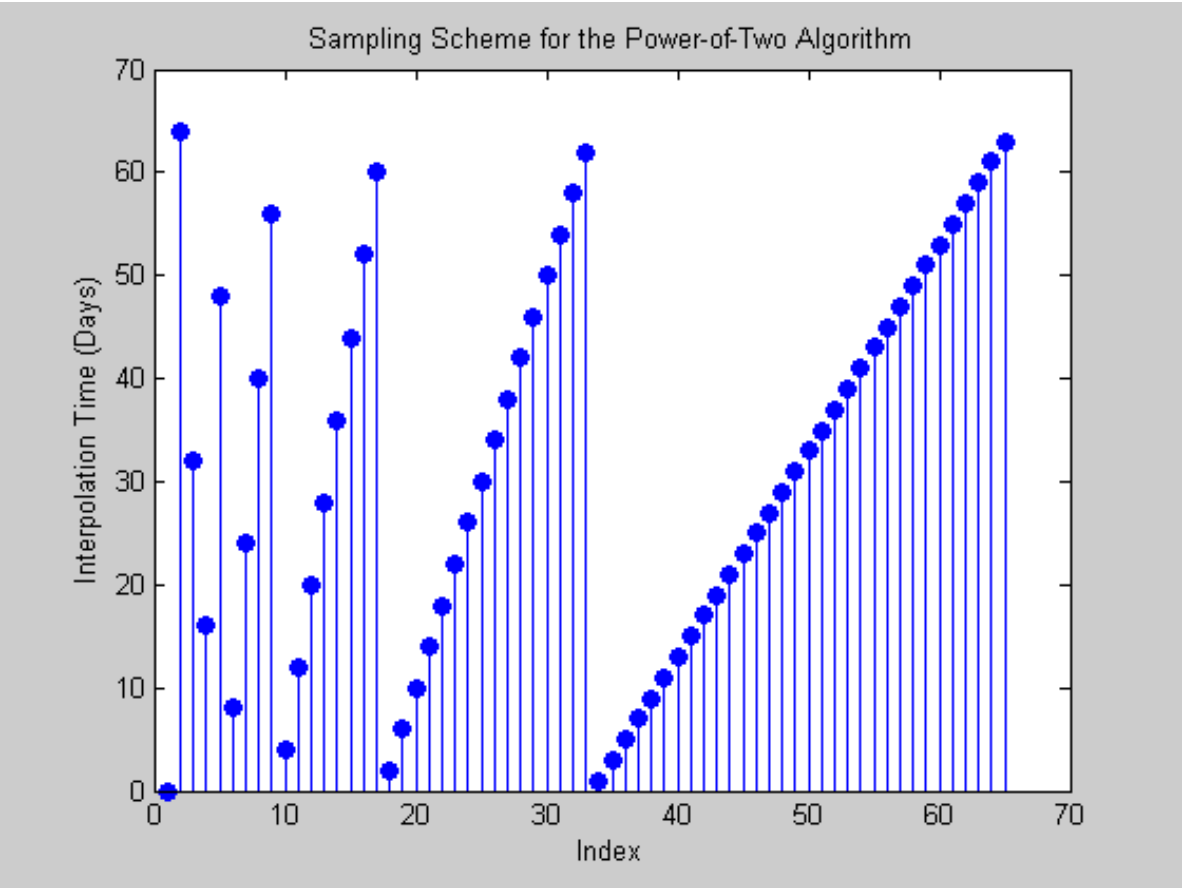
Furthermore, assume that the initial time and state coincide with those of the last available observation of the historical data, and the terminal state is the expected value of the Vasicek model 64 days into the future. In this case, we may assess the behavior of various paths that all share the same initial and terminal states, perhaps to support pricing path-dependent interest rate options over a 3-month interval.

The following code segment creates a vector of interpolation times to traverse the time grid by moving both forward and backward in time. Specifically, the first interpolation time is set to the most recent short rate observation time, the second interpolation time is set to the terminal time, then subsequent interpolation times successively sample intermediate states.

```
T      = 64;                          % Terminal time in 64 days (periods)
times  = (1:T)';                      % Sample times re-arranged by the bridge
t      = NaN(length(times) + 1, 1); % Pre-allocate the interpolation times
t(1)   = obj.StartTime;               % 1st time = last observation time
t(2)   = T;                           % 2nd time = terminal time

delta  = T;
jMax   = 1;
iCount = 3;

for k = 1:log2(T)
    i = delta / 2;
    for j = 1:jMax
        t(iCount) = times(i);
        i         = i + delta;
        iCount    = iCount + 1;
    end
    jMax  = 2 * jMax;
    delta = delta / 2;
end
```

It is instructive to examine the sequence of interpolation times generated by this popular algorithm,

```
stem(1:length(t), t, 'filled')
xlabel('Index'), ylabel('Interpolation Time (Days)')
title ('Sampling Scheme for the Power-of-Two Algorithm')
```



Notice that the first few samples are widely-separated in time and determine the course structure of the paths, while later samples are closely-spaced and progressively refine the detailed structure.

Now that the sequence of interpolation times has been generated, initialize a course time series grid to initiate the interpolation. The sampling process begins at the last observed time and state taken from the historical short rate series, and ends 64 days into the future at the expected value of the Vasicek model derived from the calibrated parameters.

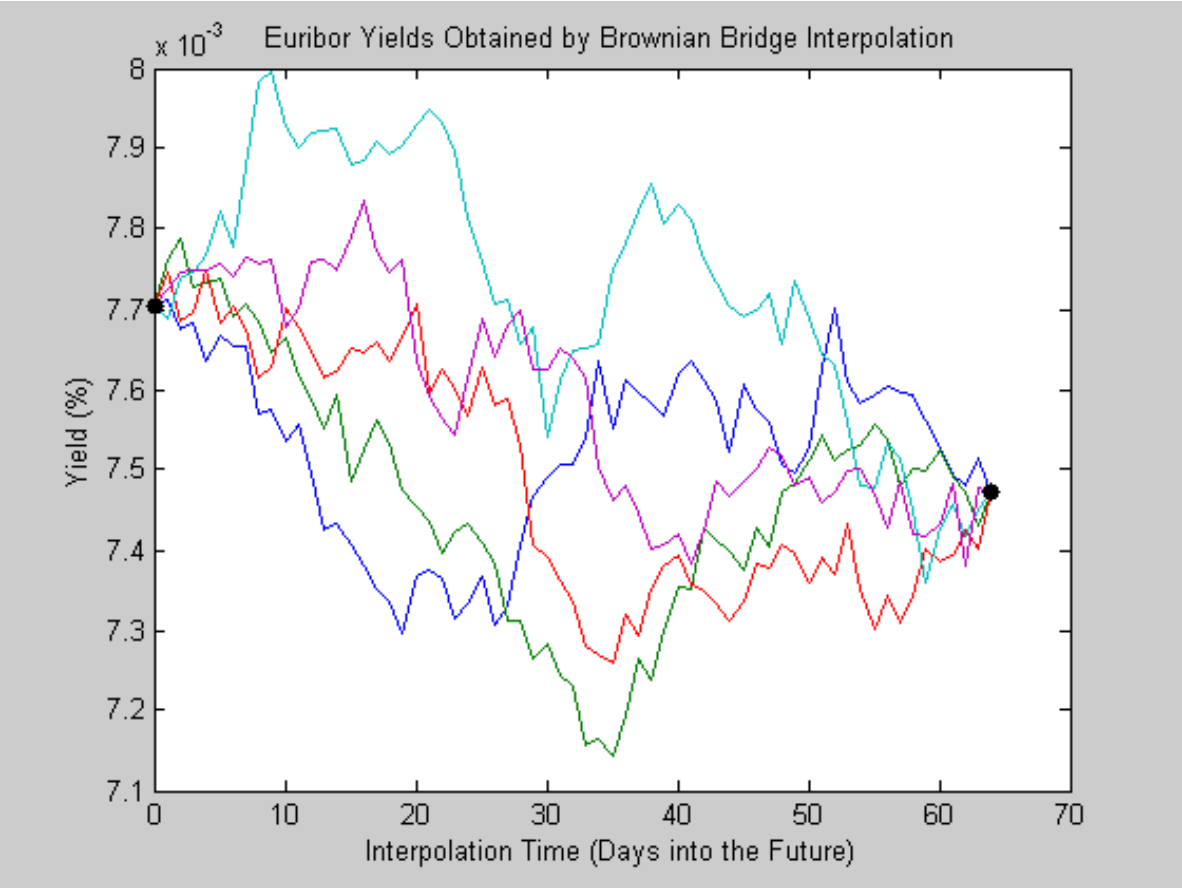
```
average = obj.StartState * exp(-speed * T) + level * (1 - exp(-speed * T));
X       = [obj.StartState ; average];
```

Now generate 5 sample paths setting the Refine input flag to TRUE, indicating that each new interpolated state is inserted into the time series grid as it becomes available. Notice that interpolation is performed on a trial-by-trial basis, and since the input time series X has 5 trials (each page of the 3-D time series represents an independent trial), the interpolated output series Y has 5 pages too.

```
nTrials = 5;
randn('state', 0)

Y = obj.interpolate(t, X(:,:,ones(nTrials,1)), 'Times' , [obj.StartTime  T], ...
                                               'Refine', true);
```

Finally, plot the resulting sample paths. Since the interpolation times are not monotonically increasing, sort the times and re-order the corresponding short rates.

```
[t,i] = sort(t);
Y     = squeeze(Y);
Y     = Y(i,:);
plot(t, 100 * Y), hold('on')
plot(t([1 end]), 100 * Y([1 end],1), '. black', 'MarkerSize', 20)
xlabel('Interpolation Time (Days into the Future)')
ylabel('Yield (%)'), hold('off')
title ('Euribor Yields Obtained by Brownian Bridge Interpolation')
```



The short rates shown above represent alternative sample paths that share the same initial and terminal values, and illustrate a special, albeit simplistic, case of a broader sampling technique known as stratified sampling. For a more sophisticated example of stratified sampling, see *Example: User-Specified Random Number Generation Part II: Stratified Sampling*.

Although this simple example simulated a univariate Vasicek interest rate model, it is applicable to problems of any dimensionality.

Example: End-of-Period Processes: Black-Scholes Option Pricing

All simulation and interpolation methods allow users to specify a sequence of functions, or background processes, to evaluate at the end of every sample time. These functions are specified as callable functions of time and state, and must return an updated state vector X,

$$X_t = f(t, X_t)$$

If more than one processing function is specified, they must be entered as a cell array of functions, and are

invoked in the order found in the cell array.

Notice that processing functions are not obligated to use time (t) or state (X), nor are they required to update or change the input state vector in any way. In fact, simulation/interpolation methods have no knowledge of any implementation details, and in this respect only adhere to a published interface.

At first glance, the presence of such processing functions may seem strange, yet they offer users a powerful modeling tool with applications limited only by the imagination. Such functions allow users to, for example, specify boundary conditions, accumulate statistics, plot graphs, and price path-dependent options.

Moreover, processing functions also allows users to avoid simulation outputs altogether.

As an example, consider pricing a European stock option by Monte Carlo simulation within a Black-Scholes-Merton framework. Assume the following characteristics:

- stock is currently trading at 100
- stock pays no dividends
- stock volatility is 50% per annum
- option strike price is 95
- option expires in 3 months
- risk-free rate is constant at 10% per annum

To solve this problem, model the evolution of the underlying stock by a univariate geometric Brownian motion (GBM) model with constant parameters,

$$dX_t = 0.1X_t dt + 0.5X_t dW_t$$

Furthermore, assume we are interested in simulating the stock price on a daily basis, and that each calendar month is composed of 21 trading days.

```
strike   = 95;             % exercise price
rate     = 0.1;            % annualized risk-free rate
sigma    = 0.5;            % annualized volatility
dt       = 1 / 252;        % time increment = 1 day = 1/252 years
nPeriods = 63;             % # of simulation periods in 3 months = 63 trading days
T        = nPeriods * dt;  % time to expiration = 3 months = 0.25 years

obj = gbm(rate, sigma, 'StartState', 100);
```

The goal is to simulate independent paths of daily stock prices, and calculate the price of European options as the risk-neutral sample average of the discounted terminal option payoff at expiration 63 days from now. This example calculates option prices by two approaches.

The first approach performs a traditional Monte Carlo simulation, explicitly requesting the simulated stock paths as an output. These output paths are then post-processed to price the options.

The second approach specifies an end-of-period processing function, accessible by time and state, which records the terminal stock price of each sample path. This processing function is implemented as a nested function with access to shared information,

```
clc, type blackScholesExample
```

```
function f = blackScholesExample(nTimes, nPaths)
%Black-Scholes European Option Pricing Monte Carlo Example.
%   Demonstrate a simple end-of-period processing function to price European
%   options by Monte Carlo simulation assuming a constant risk-free rate. The
%   example also illustrates how to update, access, and share information among
%   several nested functions.
%
%   f = blackScholesExample(nTimes, nPaths)
%
% Inputs:
%   nTimes - Number of simulation times steps (nTimes = nPeriods * nSteps).
%   nPaths - Number of independent sample paths (i.e., simulation trials).
%
% Output:
%   f - Structure of nested function handles.

% Copyright 1999-2007 The MathWorks, Inc.
% $Revision$   $Date$

prices = zeros(nPaths,1);                % Pre-allocate terminal price vector.
iTime  = 0;                              % Counter for the period index.
iPath  = 1;                              % Counter for the trial  index.
T      = 0;                              % Initialize time-to-expiry.
tStart = 0;                              % Initialize sample time.

f.BlackScholes = @saveTerminalStockPrice;  % End-of-period processing function.
f.CallPrice    = @getCallPrice;            % Call option pricing utility.
f.PutPrice     = @getPutPrice;             % Put option pricing utility.

function X = saveTerminalStockPrice(t,X)   % The actual workhorse, f(t,X).
  if iTime == 0                            % Account for initial validation.
     iTime  = 1;
     tStart = t;
  else                                     % Simulation/interpolation is live!
     if iTime < nTimes
      % Do something here if necessary.
        iTime = iTime + 1;                 % Update the period counter.
     else                                  % The last period of the current trial.
        prices(iPath) = X;                 % Save the terminal price for this trial.
        iTime = 1;                         % Re-set the time period counter.
        iPath = iPath + 1;                 % A new trial has begun.
        T     = t - tStart;                % Accumulate the time-to-expiration.
     end
  end
end

function value = getCallPrice(strike, rate)
  value = mean(exp(-rate * T) * max(prices - strike, 0));
end

function value = getPutPrice(strike, rate)
  value = mean(exp(-rate * T) * max(strike - prices, 0));
```

```
end

end  % End of outer/primary function.
```

Before simulation, first invoke the example file to access the end-of-period processing function,

```
clc
nTrials = 2000;            % # of independent trials (i.e., paths)
f       = blackScholesExample(nPeriods, nTrials)
```

```
f =
    BlackScholes: @blackScholesExample/saveTerminalStockPrice
       CallPrice: @blackScholesExample/getCallPrice
        PutPrice: @blackScholesExample/getPutPrice
```

Now simulate independent trials (sample paths). Notice that the simulated stock price paths are requested as an output and that an end-of-period processing function is specified,

```
randn('state', 0)

X = obj.simBySolution(nPeriods, 'DeltaTime', dt, 'nTrials', nTrials, ...
                      'Processes', f.BlackScholes); clc
```

Now calculate the option prices directly from the simulated stock price paths. Since these are European options, the following code segment simply ignores all intermediate stock prices,

```
call = mean(exp(-rate * T) * max(squeeze(X(end,:,:)) - strike, 0))
put  = mean(exp(-rate * T) * max(strike - squeeze(X(end,:,:)), 0))
```

```
call =
   14.2566
put =
    6.0388
```

Now price the options indirectly by invoking the nested functions found in the structure of function handles,

```
f.CallPrice(strike, rate)
f.PutPrice (strike, rate)
```

```
ans =
   14.2566
ans =
    6.0388
```

For reference, the theoretical call and put prices computed from the Black-Scholes option formulas are 13.6953 and 6.3497, respectively.

Although the same option prices are obtained, the latter approach works directly with the terminal stock prices of each sample path, and is therefore much more memory efficient. In fact, in this example there is no compelling reason to request an output at all.

To verify this claim, perform the simulation again with no outputs. Notice that the following code segment must re-initialize/re-synchronize the nested functions by calling the **blackScholesExample** function again,

```
randn('state', 0)
f = blackScholesExample(nPeriods, nTrials);

obj.simBySolution(nPeriods, 'DeltaTime', dt, 'nTrials', nTrials, ...
                  'Processes', f.BlackScholes);

f.CallPrice(strike, rate)
f.PutPrice (strike, rate)
```

```
ans =
   14.2566
ans =
    6.0388
```

Although this is a rather simplistic example, it illustrates several important features.

First, notice that the Black-Scholes example file consists of several utility functions nested within an outer, or primary, function. The outer function is invoked with problem-specific parameters needed to initialize and pre-allocate information shared by the nested functions and retained from one call to the next.

In other words, we have effectively created an object that encapsulates information and allows users to access and manipulate this information by invoking nested functions. In fact, the same effect could have been implemented by formally designing a class and creating an object with various methods, but a simple structure with nested functions offers a convenient alternative.

In this respect, the outer function does not calculate anything per se, but rather posts shared information. The output is not option prices, but rather a data structure whose fields encapsulate the details of the nested functions. The user is then able to access, update, and manipulate this shared information by invoking nested functions, at least one of which must be accessible by the (t,X) interface.

Example: User-Specified Random Number Generation Part I: Antithetic Sampling

All simulation methods allow users to directly specify the random noise process used to generate the Brownian motion vector (Wiener process) which, in turn, drives the Monte Carlo simulation. Note that the interpolation method does **not** support user-specified noise processes.

The noise process may be specified in 2 ways:

- As an explicit 3-D time series array of dependent random variates, or

- As a callable function of time and state

When specified as a function, the noise generator must be of the form

$$z_t = Z(t, X_t)$$

whose output is column vector of (possibly dependent) random variates whose length is consistent with the dimension of the Brownian motion. If no noise process is specified, then correlated Gaussian variates are generated based on the `Correlation` member of the SDE object.

As for end-of-period processing functions, noise functions are not obligated to use time (t) or state (X). In fact, simulation methods have no knowledge of any implementation details, and in this respect only adhere to a published interface.

At first glance, the presence of noise generation functions may seem strange, yet such functions allow users to, for example, implement conditional sampling and variance reduction techniques, incorporate non-Gaussian processes, or model stochastic correlation structures, etc ...

To demonstrate user-specified noise processes, consider a popular *variance reduction* technique known as *antithetic sampling* applied to a path-dependent barrier option. Antithetic sampling is a relatively straightforward technique that attempts to replace one sequence of random observations with another of the same expected value but smaller variance. The technique attempts to reduce variance by inducing negative dependence between paired input samples in the hope that negative dependence between paired output samples will result. The greater the extent of negative dependence, the more effective antithetic sampling will be.

To demonstrate, assume we wish to value a European up-and-in call option on a single underlying stock whose price evolution is governed by geometric Brownian motion (GBM) model with constant parameters,

$$dX_t = 0.05X_t dt + 0.3X_t dW_t$$

and assume the following characteristics:

- stock is currently trading at 105
- stock pays no dividends
- stock volatility is 30% per annum
- option strike price is 100
- option expires in 3 months
- option barrier is 120
- risk-free rate is constant at 5% per annum

Furthermore, assume we are interested in simulating the stock price on a daily basis,

```
clc
barrier  = 120;           % barrier
strike   = 100;           % exercise price
rate     = 0.05;          % annualized risk-free rate
sigma    = 0.3;           % annualized volatility
nPeriods = 63;            % # of simulation periods in 3 months = 63 trading days
dt       = 1 / 252;       % time increment = 252 days = 1 year
T        = nPeriods * dt;  % time to expiration = 3 months = 0.25 years
```

```
obj = gbm(rate, @(t,X) sigma, 'StartState', 105);
```

The goal is to simulate various paths of daily stock prices, and calculate the price of the barrier option as the risk-neutral sample average of the discounted terminal option payoff. Since this is a barrier option we must also determine if and when the barrier is crossed. This example calculates the option's price using 2 completely different techniques:

- perform antithetic sampling by explicitly setting the flag `Antithetic = true`,
- specify a function, Z(t,X), to generate antithetic Gaussian samples, **and** specify an end-of-period processing function to record the maximum and terminal stock prices on a path-by-path basis

The first technique performs a Monte Carlo simulation by requesting the simulated stock paths as an output, and does so by explicitly asking for an antithetic sample. These output paths are then post-processed in pairs to price the option. For small-to-medium-sized simulations, this offers an acceptable compromise between convenience, memory efficiency, and runtime performance.

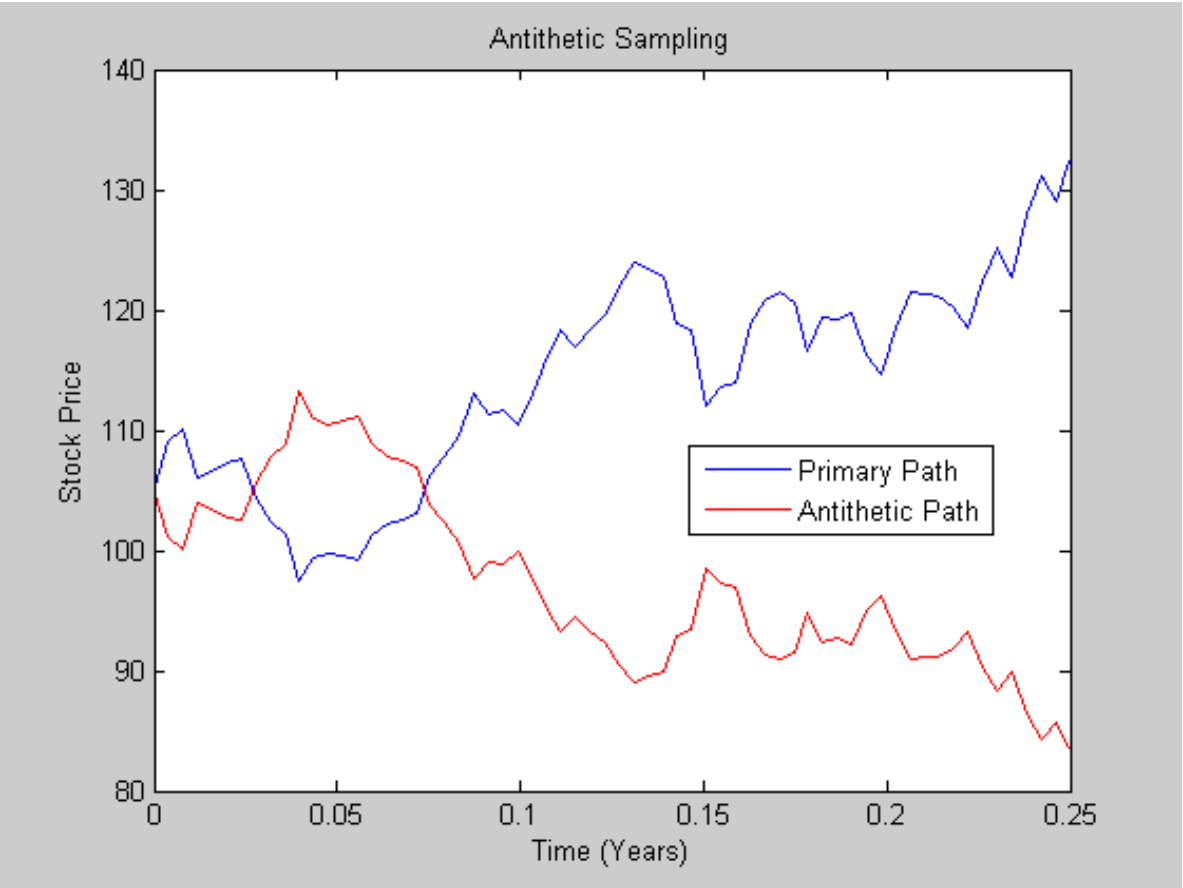
To illustrate antithetic sampling, consider a small-scale simulation designed to price the barrier option discussed above. The first technique simulates 200 paths and sets the `Antithetic` input flag to true, producing a 3-D time series matrix with 200 pages (i.e., the third dimension is 200).

When antithetic sampling is requested, the sampling is performed such that all primary and antithetic paths are simulated and stored in successive matching pairs: Odd paths (1,3,5,...) correspond to the primary Gaussian paths and even paths (2,4,6,...) are the matching antithetic paths of each pair derived by negating the Gaussian draws of the corresponding primary (odd) path.

```
nTrials = 200;          % # of simulated paths = 100 primary/antithetic pairs
randn('state', 10)
[X, Time] = obj.simBySolution(nPeriods, 'DeltaTime', dt, 'nTrials', nTrials, ...
                                        'Antithetic', true);
```

To illustrate the effect of antithetic sampling, plot the sample paths of the first primary/antithetic pair. Also notice that, when working with univariate models, it is often convenient to re-order the 3-D output path arrays as 2-D equivalents by removing the singleton second dimension (i.e., the number of columns in univariate models is 1, which is slightly awkward to manipulate).

```
X = squeeze(X);

plot(Time, X(:,1), 'blue', Time, X(:,2), 'red')
xlabel('Time (Years)'), ylabel('Stock Price'), title('Antithetic Sampling')
legend({'Primary Path' 'Antithetic Path'}, 'Location', 'Best')
```



In contrast, the second technique takes a completely different approach.

Specifically, in many situations the entire simulated process is unnecessary, which offers an opportunity to dramatically reduce memory usage or increase the scale of the simulation. The following technique exploits this opportunity by specifying a function to explicitly generate antithetic Gaussian variates on a path-by-path basis,

```
type antithetic1Example
```

```
function f = antithetic1Example(nTimes, nBrownians)
%Antithetic Sampling Random Number Generation Example.
%   Demonstrate a user-specified function to generate independent Gaussian
%   antithetic samples on a path-by-path basis. Although this function generates
%   independent Gaussian noise, arbitrary correlation could be added. This
%   example also illustrates how to access and update saved information.
%
%   f = antithetic1Example(nTimes, nBrownians)
%
% Inputs:
%   nTimes     - Number of simulation times steps (nTimes = nPeriods * nSteps).
%   nBrownians - Dimension of the Brownian motion vector.
%
% Output:
%   z - Function handle to generate independent Gaussian antithetic samples on
%       a path-by-path basis.
```

```
% Copyright 1999-2007 The MathWorks, Inc.
% $Revision$   $Date$

iTime = 0;                                  % Counter for the period index.
iPath = 1;                                  % Counter for the trial  index.
z     = randn(nBrownians,nTimes);           % Generate uncorrelated Gaussians.
f     = @antitheticSampling;                % Antithetic Gaussian noise function.

function Z = antitheticSampling(t, X)       % The actual workhorse, Z(t,X).
  if iTime == 0                             % Account for initial validation.
     iTime = 1;
     Z     = z(:,iTime);
  else                                      % Simulation is live!
     Z     = z(:,iTime);                    % Return the next Gaussian vector.
     if iTime < nTimes
        iTime = iTime + 1;                  % Update the period counter.
     else                                   % The last period of the current path.
        iTime = 1;                          % Re-set the time period counter.
        iPath = iPath + 1;                  % A new path will begin next time.
        if rem(iPath, 2) == 0               % Is this primary or antithetic path?
           z = -z;                          % Even paths are antithetic paths.
        else
           z = randn(nBrownians,nTimes);    % Odd paths are primary paths.
        end
     end
  end
end
end  % End of outer/primary function.
```

In addition, the technique also specifies an end-of-period processing function which records the maximum and terminal stock prices of each sample path. This processing function is also accessible by time and state, and is implemented as a nested function with access to shared information.

This shared information also allows the option price and corresponding standard error to be calculated, and operates in a manner similar to the Black-Scholes example already discussed (see *Example: End-of-Period Processes Part: Black-Scholes Option Pricing*).

```
clc, randn('state', 10)
z = antithetic1Example(nPeriods, 1)        % antithetic Gaussian sampling function
f = antithetic2Example(nPeriods, nTrials)  % option pricing & standard error
```

```
z =
    @antithetic1Example/antitheticSampling
f =
      SaveMaxLast: @antithetic2Example/saveMaxLastPrices
      OptionPrice: @antithetic2Example/getBarrierOptionPrice
    StandardError: @antithetic2Example/getStandardError
```

Now perform the same antithetic sampling experiment using the functions just created. Since this technique requests no outputs, it dramatically minimizes memory usage and represents a far more elegant solution,

```
obj.simBySolution(nPeriods, 'DeltaTime', dt, 'nTrials', nTrials, 'Z', z, ...
                            'Processes', f.SaveMaxLast); clc
```

Now approximate the option price and 95% confidence interval obtained from latter technique. For comparison, the value of the up-and-in barrier option is approximately 7.83, as verified by a Cox-Ross-Rubinstein binomial tree.

```
optionPrice   = f.OptionPrice  (strike, rate, barrier);
standardError = f.StandardError(strike, rate, barrier, true);
lowerBound    = optionPrice - 1.96 * standardError;
upperBound    = optionPrice + 1.96 * standardError;

fprintf('  Up-and-In Barrier Option Price: %8.4f\n', optionPrice)
fprintf('         Standard Error of Price: %8.4f\n', standardError)
fprintf(' Confidence Interval Lower Bound: %8.4f\n', lowerBound)
fprintf(' Confidence Interval Upper Bound: %8.4f\n', upperBound)
```

```
  Up-and-In Barrier Option Price:   7.4549
         Standard Error of Price:   0.6763
 Confidence Interval Lower Bound:   6.1294
 Confidence Interval Upper Bound:   8.7805
```

Example: User-Specified Random Number Generation Part II: Stratified Sampling

As discussed in *Example: User-Specified Random Number Generation Part I: Antithetic Sampling*, all simulation methods allow users to directly specify a noise process as a callable function of time and state,

$$z_t = Z(t, X_t)$$

Similar to antithetic sampling, *stratified sampling* is also a variance reduction technique, yet operates by constraining a proportion of sample paths to specific subsets, or *strata*, of the sample space rather than inducing negative dependence between paired samples.

Specifically, this example specifies a noise function to stratify the terminal value of a univariate equity price series. Starting from known initial conditions, the function first stratifies the terminal value of a standard Brownian motion, then samples the process from beginning to end by drawing conditional Gaussian samples via a Brownian bridge.

The stratification assumes that each path is associated with a single stratified terminal value such that the number of paths is equal to the number strata, a technique referred as *proportional sampling*. This example is similar to that found in *Example: The Brownian Bridge Part II: Stochastic Interpolation with Refinement*, yet more sophisticated.

Since stratified sampling requires knowledge of the future, it also requires more sophisticated time synchronization. Specifically, the example function requires knowledge of the entire sequence of sample times,

```
clc, type stratifiedExample
```

```
function z = stratifiedExample(nPaths, times)
%Univariate Terminal Stratification Random Number Generation Example.
%   Demonstrate a user-specified noise function to stratify the terminal value
%   of a standard Brownian motion. The process is then sampled from beginning to
%   end by drawing conditional Gaussian samples via a Brownian bridge.
%
%   z = antithetic1Example(nPaths, Times)
%
% Inputs:
%   nPaths - Number of sample paths (the same as nTrials in this example).
%   Times  - Vector of sample times associated with all simulated paths.
%
% Output:
%   z - Function to generate stratified Gaussian samples on a path-by-path
%     basis. Stratification assumes that each path is associated with a single
%     stratified terminal value such that the number of paths is equal to the
%     number strata (proportional sampling). Note that the noise function does
%     not return the actual Brownian paths, but rather the N(0,1) Gaussian draws
%     Z(t,X) that drive it.

% Copyright 1999-2007 The MathWorks, Inc.
% $Revision$   $Date$

iTime  = 0;                                % Counter for the period index.
iPath  = 1;                                % Counter for the trial  index.
W1     = 0;                                % Initialize the Brownian state.
T      = times(end);                       % Terminal (final) sample time.
dt     = diff(times(:));                   % Time increments.
nTimes = numel(dt);                        % # of time increments per sample path.

U  = ((1:nPaths)' - 1 + rand(nPaths,1)) / nPaths;
WT = norminv(U) * sqrt(T);                 % Stratify terminal Brownian motion.
z  = @stratifiedSampling;                  % Brownian bridge stratification function.

function Z = stratifiedSampling(t, X)      % The actual workhorse, Z(t,X).
  if iTime == 0                            % Account for initial validation.
     iTime = 1;
     Z     = 0;
  else                                     % Simulation is live!
     tInsert  = t + dt(iTime);             % The interpolation time.

     drift    = ((T - tInsert) * W1  +  dt(iTime) * WT(iPath)) / (T - t);
     variance =  (T - tInsert) * dt(iTime) / (T - t);
     W2       = drift  +  sqrt(variance) * randn(1);

     dW       = W2 - W1;                   % Brownian increment.
     Z        = dW / sqrt(dt(iTime));      % Return the next Gaussian vector.
     W1       = W2;                        % Save the previous Brownian state.

     if iTime < nTimes
        iTime = iTime + 1;                 % Update the period counter.
     else                                  % The last period of the current path.
        iTime = 1;                         % Re-set the time period counter.
```

```
        iPath = iPath + 1;                % A new path will begin next time.
        W1    = 0;
      end
    end
  end
  end  % End of outer/primary function.
```

The function implements proportional sampling by partitioning the unit interval into bins of equal probability by first drawing a random number uniformly distributed in each bin. These stratified uniform draws are then transformed by the inverse cumulative distribution function of a standard N(0,1) Gaussian distribution. Finally, the resulting stratified Gaussian draws are scaled by the square root of the terminal time to stratify the terminal value of the Brownian motion.

Notice that the noise function above does **not** return the actual Brownian paths, but rather the Gaussian draws Z(t,X) that drive it. Since the simulation method will scale Z(t,X) by the square root of the current time increment `dt`, the function divides by the square root of `dt`.

To introduce the sampling technique, suppose we wish to stratify the terminal value of a univariate, zero-drift, unit-variance-rate Brownian motion (BM) model,

$$dX_t = dW_t$$

Furthermore, assume we are interested in simulating 10 paths of the process on a daily basis over a 3 month period, and that each calendar month and year is composed of 21 and 252 trading days, respectively.,

```
clc, randn('state', 10), rand('state', 0)

dt       = 1 / 252;        % time increment = 1 day = 1/252 years
nPeriods = 63;             % # of simulation periods in 3 months = 63 trading days
T        = nPeriods * dt;  % time to expiration = 3 months = 0.25 years
nPaths   = 10;             % # of simulated paths

obj         = bm(0, 1, 'StartState', 0);
sampleTimes = cumsum([obj.StartTime ; dt(ones(nPeriods,1))]);
z           = stratifiedExample(nPaths, sampleTimes)
```

```
z =
    @stratifiedExample/stratifiedSampling
```

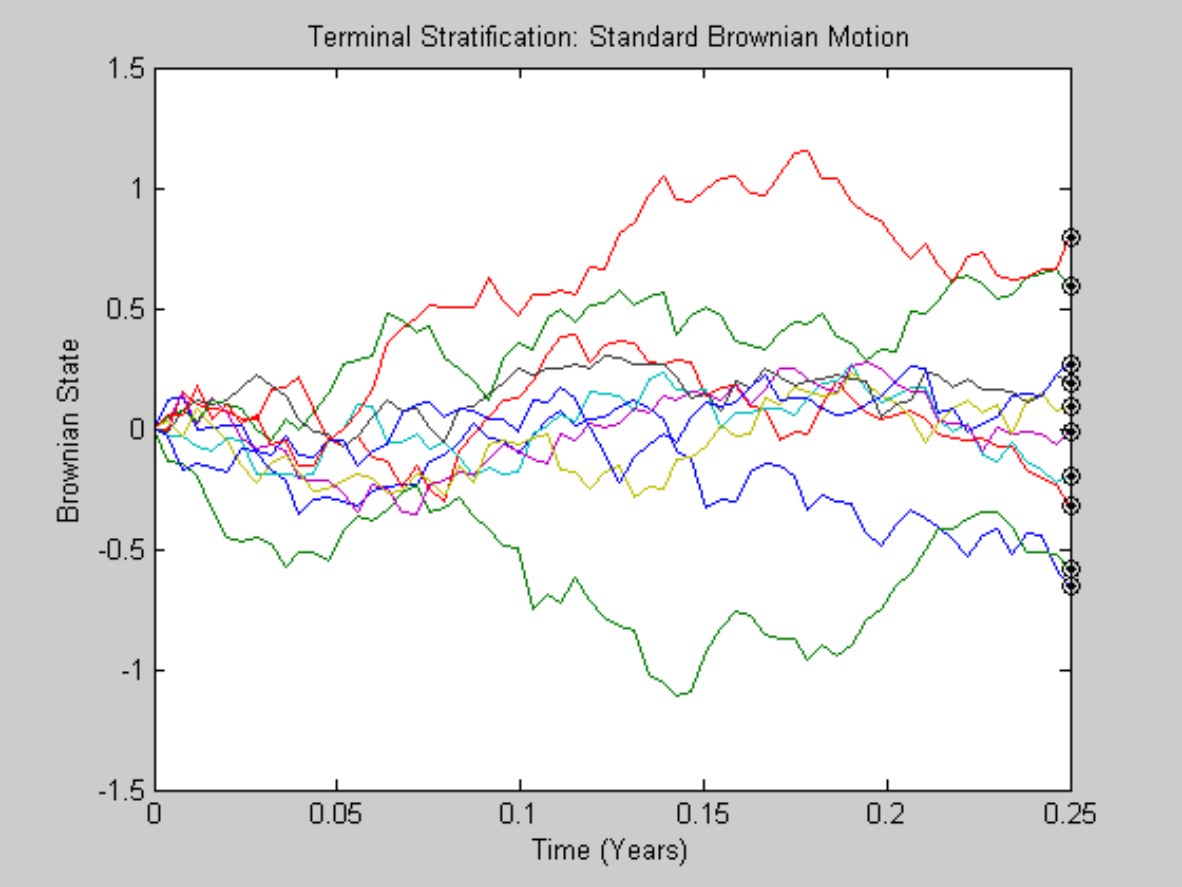
Now simulate the standard Brownian paths by explicitly passing the stratified sampling function to the simulation method, and re-order the output sample paths for convenience,

```
X = obj.simulate(nPeriods, 'DeltaTime', dt, 'nTrials', nPaths, 'Z', z);
X = squeeze(X); % Re-order the 3-D output to a 2-D equivalent array.
```

To verify the stratification, re-create the uniform draws with proportional sampling, transform them to obtain the terminal values of standard Brownian motion, and plot the terminal values and output paths on

the same figure,

```
rand('state', 0)                                    % Use the same initial state.
U  = ((1:nPaths)' - 1 + rand(nPaths,1))/nPaths;  % Stratified uniforms in each bin.
WT = norminv(U) * sqrt(T);                          % Stratified Brownian motion.

plot(sampleTimes, X), hold('on')
xlabel('Time (Years)'), ylabel('Brownian State')
title('Terminal Stratification: Standard Brownian Motion')
plot(T, WT, '. black', T, WT, 'o black')
hold('off')
```



Notice that the last value of each sample path (i.e., the last row of the output array X) exactly coincides with the corresponding element of the stratified terminal value of the Brownian motion. This occurs because the simulated model and the noise generation function both represent the same standard Brownian motion.

However, the same stratified sampling function may also be used to stratify the terminal price of constant-parameter geometric Brownian motion models. In fact, the stratified sampling function may be used to stratify the terminal value of any constant-parameter model driven by Brownian motion, provided the model's terminal value is a monotonic transformation of the terminal value of the Brownian motion.

To illustrate this, assume we wish to simulate risk-neutral sample paths of the FTSE 100 index using a geometric Brownian motion (GBM) model with constant parameters,

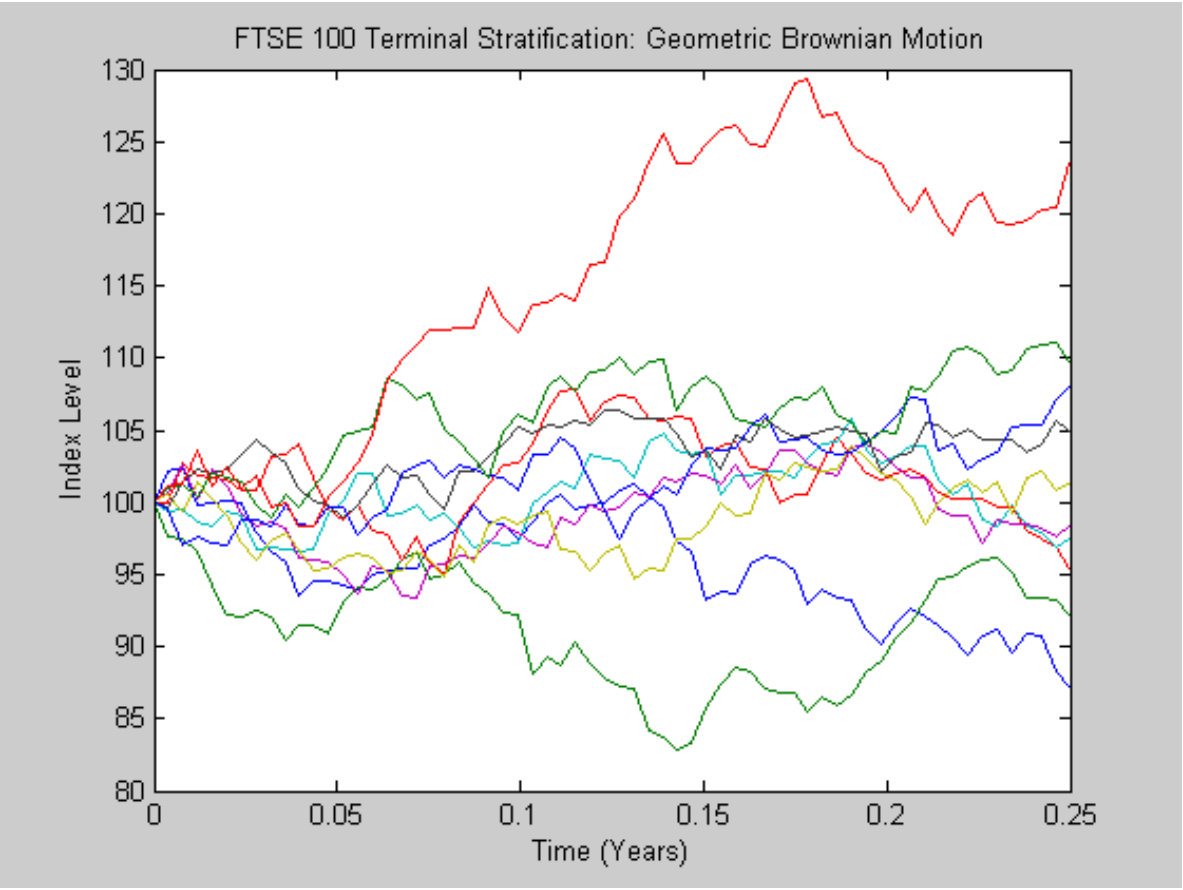
$$dX_t = rX_t dt + \sigma X_t dW_t$$

in which Euribor yields represent the risk-free rate of return. Furthermore, assume we wish to annualize the relevant information derived from the daily data, and that each calendar year is composed of 252 trading days.

```
returns = price2ret(SDE_Data.UK);      % daily log returns of FTSE 100
sigma   = std(returns) * sqrt(252);    % annualized volatility
rate    = SDE_Data.Euribor3M;
rate    = mean(360 * log(1 + rate));   % continuously-compounded, annual yield
```

Now create the GBM model assuming the FTSE 100 starts at 100, determine the sample times, and simulate the price paths,

```
obj         = gbm(rate, sigma, 'StartState', 100);
nSteps      = 1;
sampleTimes = cumsum([obj.StartTime ; dt(ones(nPeriods * nSteps,1))/nSteps]);
z           = stratifiedExample(nPaths, sampleTimes);

randn('state', 10), rand('state', 0)
[Y, Times] = obj.simBySolution(nPeriods  , 'nTrials'   , nPaths, ...
                               'DeltaTime', dt, 'nSteps', nSteps,  'Z', z);
Y = squeeze(Y);   % Re-order the 3-D output to a 2-D equivalent array.

figure, plot(Times, Y)
xlabel('Time (Years)'), ylabel('Index Level')
title('FTSE 100 Terminal Stratification: Geometric Brownian Motion')
```



Although the terminal value of the Brownian motion shown in the first graph is normally-distributed, and the terminal price in the second graph is lognormally-distributed, notice the similarity between the corresponding paths of each graph.

As in the antithetic sampling example, this example could avoid outputs entirely provided the user specifies an end-of-period processing function to query the terminal state, thereby dramatically reducing memory usage.