An Incomplete Market Approach to Employee Stock Option Valuation

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MODERN TOOLS FOR FINANCIAL ANALYSIS AND MODELING

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An Employee Stock Option

An Employee Stock Option (ESO):

- A call option on firm’s stock granted by a firm to its employees as a benefit in addition to the salary
- Popular in the US: 94% of S&P500 grant ESOs to its top executives
- “Fair” valuation required by accounting standards (IFRS 2, FAS 123R)

Employee Stock Options versus Standard Options

ESOs differ from standard (complete market) call options:

1. **ESO holders are not allowed to sell ESOs (and short the underlying stock)**
2. **ESOs have a vesting period during which they cannot be exercised. After vesting American type options**
3. **If the job of an ESO holder is terminated, his ESOs:**
    a) **forfeits if unvested**
    b) **must be exercised immediately if vested**
4. **ESOs are long-term options (maturity up to 10 years)**

ESOs Introduce Incomplete Markets

The trading restrictions (1.) and the job termination risk (3.) imply that risk-averse ESO holders exercise earlier than the risk-neutrality dictates for standard options:

1. **ESOs should be less valuable than standard options**
2. **The firm is exposed to possible hedging errors when replicating the ESO payoff, which introduces an incomplete market**

How to calculate the “fair” ESO value from the perspective of the firm?

What are the ESO costs to shareholders?

How to hedge an ESO?

The Czech Way: ČEZ ESO Programs

ČEZ a.s.:

- The largest electricity producer in the Czech Republic.
- The biggest market capitalization on Prague Stock Exchange ($20B).
- 70% state owned.
- A controversial ESO program for its top executives since 2001.
- The CEZ CEO cashed in over $40millions during his 4-year tenure.

The CEZ ESO program was redesigned twice. In 05/2006 a payoff cap imposed:

	Program-01 before 05/2006	Program-06 05/2006–05/2008	Program-08 after 05/2008
Instrument	ESO-01	ESO-06	ESO-08
Granting year	0	0	0, 1, 2, 3
Exercise price	6-month average	1-month average	1-month average
Maturity	4.25 years	5 years	3.5 years
First exercise	3 months	1,2,3 years, always 1/3	2 years
Vesting	3 months	1,2,3 years, always 1/3	1 year
Termination	3 months	1 year	remaining life
$g^+(S_\tau) =$	$\{S_\tau - K\}^+$	$\{\min\{S_\tau, 2K\} - K\}^+$	$\{\min\{S_\tau, 2K\} - K\}^+$

Exercise Behavior of CEZ Top Executives

CEZ ESO granting and exercising in time (Program-01, CEO and Board members)

Literature Review: Approaches to ESO Valuation

Two general approaches to the early exercise and to the ESO valuation:

1. **An exogenous Poisson process for the early exercise: Jennergren and Näslund (1993), Carpenter (1998), Carr and Linetsky (2000), and many others**
2. Endogenously modeled exercise policy by utility maximization (the job termination risk still exogenous): Kulatilaka and Marcus (1994), Leung and Sircar (2009), Carpenter, Stanton and Wallace (2010).

The ESO Valuation in a Poisson Process Framework

- An ESO is liquidated (exercised or forfeited) at a random time τ by the first jump of the Poisson process with intensity λ
- The crucial assumption made in the literature: The risk of Poisson process can be diversified away.
- Therefore, we have the standard setting: complete market, risk-neutral measure Q, i.e., the B&S framework with GBM for the stock price
- Jennergren and Näslund (1993) model:

$$C(T,s)=\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau\geq T\}}e^{-rT}F(S_T)\middle|S_0=s\right]+\mathbb{E}^{\mathbb{Q}}\left[1_{\{t_v\leq\tau<T\}}e^{-r\tau}F(S_\tau)\middle|S_0=s\right]$$

where $F(S)$ is the payoff, e.g., $\max(S-K,0)$, T is the maturity, t_v is the vesting date, and r is the risk-free (discount) rate

- Since τ is exponentially distributed we get:

$$C(T,s)=\mathbb{E}^{\mathbb{Q}}\left[e^{-(\lambda+r)T}F(S_T)\middle|S_0=s\right]+\mathbb{E}^{\mathbb{Q}}\left[\int_{t_v}^{T}\lambda e^{-(\lambda+r)u}F(S_u)du\middle|S_0=s\right]$$

Can Diversification be Used?

- The Poisson jump (ESO liquidation) comes as a sudden surprise
- Thus, the crucial assumption of the complete market risk-neutral valuation is that the risk of the Poisson jump can be diversified away
- We argue that diversification is not very realistic:
    1. Too few ESOs granted to rely on the Law of Large Numbers,
    2. ESO holders exercise together, thus the Poisson processes are not independent (see ČEZ exercise patterns)
- The ESO payoff cannot be hedged perfectly, and the market is incomplete

An Incomplete Market Approach to ESO Valuation

We suggest the following objective:

- **An ESO granting firm minimizes the expected squared hedging error with respect to the Poisson jump. The objective function:**

$$J_{T,x,s}(\pi) = \mathbb{E}\left[\mathbf{1}_{\{\tau\geq T\}}\left[e^{-rT}\left(X_T - F(S_T)\right)\right]^2 \middle| X_0 = x, S_0 = s\right] + \mathbb{E}\left[\mathbf{1}_{\{t_v\leq\tau<T\}}\left[e^{-r\tau}\left(X_\tau - F(S_\tau)\right)\right]^2 \middle| X_0 = x, S_0 = s\right] + \mathbb{E}\left[\mathbf{1}_{\{0\leq\tau<t_v\}}\left[e^{-r\tau} X_\tau\right]^2 \middle| X_0 = x, S_0 = s\right],$$

  where X is the value of the hedging portfolio with an initial capital *x*

- **The hedging portofolio is required to be self-financing:**

$$dX_t = \left[rX_t + \pi_t(\mu - r)\right]dt + \pi_t\sigma dW_t,$$

  where π is the nominal amount invested into the stock, μ and σ are the stock drift and volatility, respectively

- **A version of Mean-Variance hedging problem (see Schweizer (2010) for a survey)**

ESO Hedging in Discrete-Time Economy

- **Stock price process:** $S_k = R_k S_{k-1}$, **where** $R_1, \dots, R_k$ **IID stock returns**
- **Expected excess return:** $\bar{\mu} = \mathbb{E}^{\mathbb{P}}\left[\bar{R}_{k+1}\right] = \mathbb{E}^{\mathbb{P}}\left[R_{k+1}\right] - R_{\mathrm{f}}$
- **"Volatility" of excess return:** $\bar{\sigma}^2 = \mathbb{E}^{\mathbb{P}}\left[\bar{R}_{k+1}^2\right] = \mathbb{E}^{\mathbb{P}}\left[(R_{k+1} - R_{\mathrm{f}})^2\right]$
- **Risk-free asset:** $B_k = R_{\mathrm{f}} B_{k-1}$
- **Self-financing portfolio:** $X_{k+1} = X_k R_{\mathrm{f}} + \Delta_k S_k (R_{k+1} - R_{\mathrm{f}})$,
  where Δ_k is the number of stocks at period *k*

- **Objective function (minimizing the expected squared hedging error):**

$$J_{N,x,s}(\Delta_0, \dots, \Delta_{N-1}) = \mathbb{E}\Bigg[\sum_{i=0}^{N-1} \varrho(1-\varrho)^i R_{\mathrm{f}}^{-2i}\left(X_i - F(S_i)\mathbb{1}_{\{N_v < i\}}\right)^2 + (1-\varrho)^N R_{\mathrm{f}}^{-2N}\left(X_N - F(S_N)\right)^2 \Bigg| X_0 = x, S_0 = s\Bigg],$$

where *ρ* is the probability that the ESO is liquidated during a given time period, *N* is the ESO maturity

- **Expectation taken under the objective, not risk-neutral measure**

Solving the ESO Hedging by Dynamic Programming

- **Value function:**

$$V(N,x,s) = \inf_{\Delta_0,\Delta_1,\ldots,\Delta_{N-1}} J_{N,x,s}(\Delta_0,\ldots,\Delta_{N-1})$$ **eq. (1)**

- **Bellman's principle of optimality leads to the recursive equations:**

$$V(N-k,x,s) = \min_{\Delta_k} \Big\{ \varrho\big(x - F(s)1_{\{N_v<k\}}\big)^2 + (1-\varrho)R_f^{-2}\mathbb{E}\Big[V(N-k-1,X_{k+1},S_{k+1})\Big|X_k = x, S_k = s\Big]\Big\}$$ **eq. (2)**

- **Noting that** $V(0,x,s) = \big(x - F(s)\big)^2 = x^2 - 2F(s)x + F^2(s)$

we look for a solution to eq. (2) of the form:

$$V(N-k,x,s) = f(N-k)x^2 + g(N-k,s)x + h(N-k,s),$$ **eq. (3)**

where *f, g* and *h* are appropriate functions that satisfy the initial conditions:

$$f(0) = 1, \quad g(0,s) = -2F(s), \quad h(0,s) = F^2(s)$$ **eq. (4)**

The Optimal ESO Hedging

- **Substituting expression given by eq. (3) for *V(N-k-1,·,·)* into eq. (2) and minimizing over Δ_k, we can see that**

$$V(N-k,x,s) = f(N-k)x^2 + g(N-k,sR_k)x + h(N-k,sR_k),$$

where

$$f(N-k) = \varrho + (1-\varrho)\left(1-\frac{\overline{\mu}^2}{\overline{\sigma}^2}\right)f(N-k-1),$$ **eq. (5-7)**

$$g(N-k,s) = -\varrho 2F(s)\mathbb{1}_{\{N_v<k\}} + \frac{1-\varrho}{R_\mathrm{f}}\left(\mathbb{E}[g(N-k-1,sR_{k+1})] - \mathbb{E}[g(N-k-1,sR_{k+1})\overline{R}_{k+1}]\frac{\overline{\mu}}{\overline{\sigma}^2}\right),$$

$$h(N-k,s) = \varrho F^2(s)\mathbb{1}_{\{N_v<k\}} + \frac{1-\varrho}{R_\mathrm{f}^2}\left(\mathbb{E}[h(N-k-1,sR_{k+1})] - \frac{\left(\mathbb{E}[g(N-k-1,sR_{k+1})\overline{R}_{k+1}]\right)^2}{4\overline{\sigma}^2 f(N-k-1)}\right)$$

- **Easy to implement as a computer program, ideally in Matlab!**
- **In particular, the optimal self-financing hedging strategy is Markovian and is defined as:**

$$\Delta_k^* = \Delta_k^\circ(X_k^*,S_k) \quad \text{where} \quad \Delta_k^\circ(x,s) = -\left(\frac{\overline{\mu}}{\overline{\sigma}^2 s}R_\mathrm{f}x + \frac{\mathbb{E}\left[\overline{R}_{k+1}g(N-k-1,sR_{k+1})\right]}{2\overline{\sigma}^2 f(N-k-1)s}\right)$$ **eq. (8)**

The Optimal ESO Hedging

Proposition: ***The value function defined by eq. (1), i.e.,***

$$V(N,x,s)=\inf_{\Delta_0,\Delta_1,\ldots,\Delta_{N-1}}\mathbb{E}\Big[\sum_{i=0}^{N-1}\varrho(1-\varrho)^iR_{\mathrm{f}}^{-2i}\big(X_i-F(S_i)\mathbf{1}_{\{N_v<i\}}\big)^2 + (1-\varrho)^NR_{\mathrm{f}}^{-2N}\big(X_N-F(S_N)\big)^2\Big|X_0=x,S_0=s\Big]$$

is given by

$$V(N,x,s)=f(N)x^2+g(N,s)x+h(N,s),$$

where the functions f, g, and h solve the recursive equations (5-7) with initial conditions (4). Furthermore, the optimal self-financing hedging portfolio strategy (Δ_k^*) is given in a feedback form by eq. (8).

(Proof by the standard discrete-time stochastic control theory (e.g. Bertsekas and Shreve, 1978), and the calculations outlined above.)

The Continuous-Time Model

- A continuous-time version of the mean-variance optimal ESO hedging solved by a HJB equation and “confirmed” by proving the Verification Theorem
- The value function can be separated as in the discrete-time case, and leads to PDEs for f, g, and h, which have a similar structure as in the discrete-time
- An analytical solution for the infinite horizon problem, i.e., for an ESO with infinite maturity – very tedious calculations, nice to have the Matlab Symbolic Toolbox

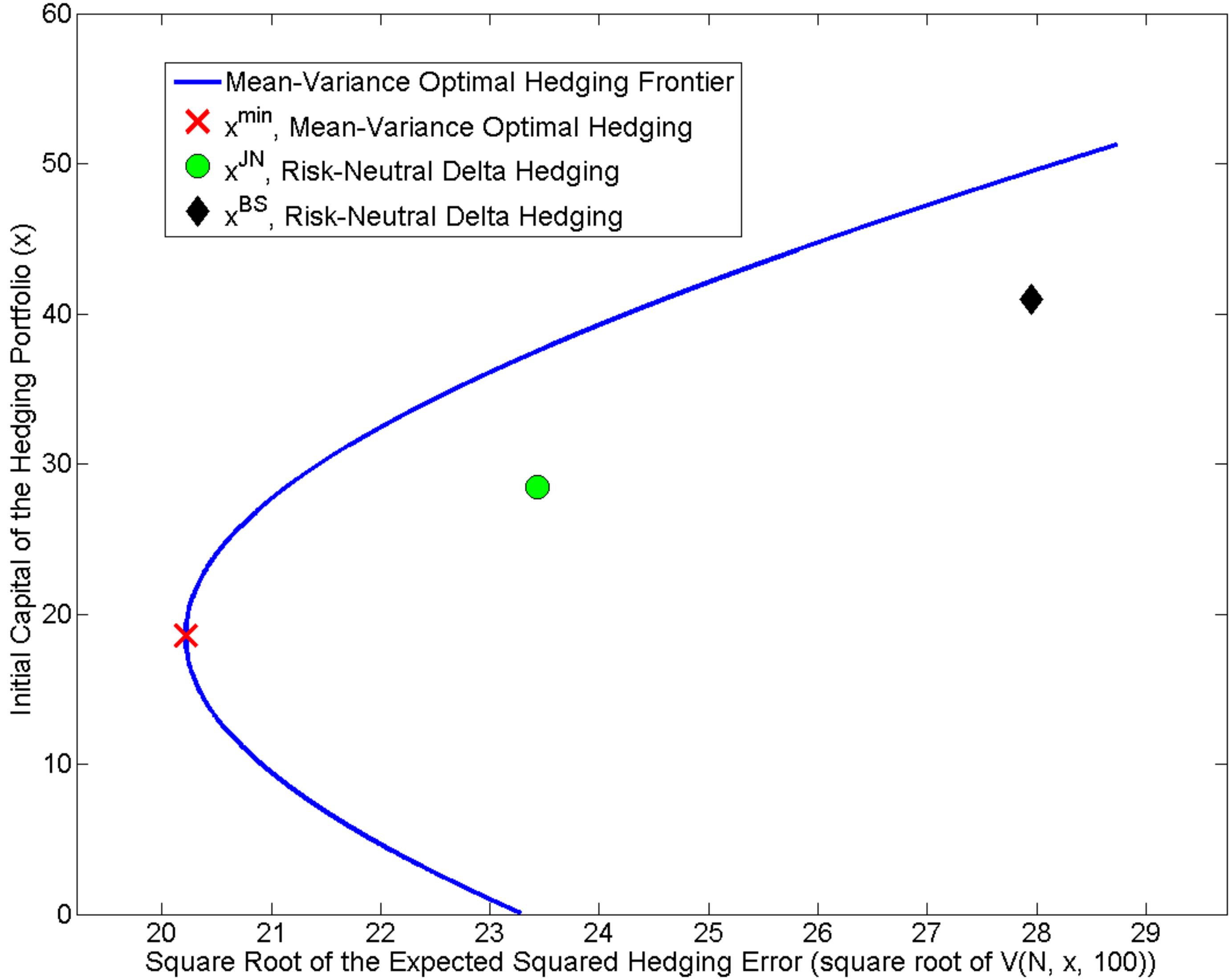
An Illustration of the Optimal ESO Hedging

- The stock S: drift $\mu = 0.12$, volatility $\sigma = 0.20$, risk-free rate R_f-1 = 0.04
- The ESO: maturity $T = 10$Y, vesting $T_v = 3$Y, $F(S) = \max(S-K,0)$, $S_0 = 100$, $K = 100$
- The Poisson process for the ESO liquidation: $\lambda = 0.08$, i.e., $P(\tau > T) = 0.45$
- Implemented on a binomial lattice
- We are interested in the smallest value function with respect to the initial capital:

$$x^{\min} = -\frac{g(N,s)}{2f(N)}, \quad V(N, x^{\min}, s) = -\frac{g^2(N,s)}{4f(N)} + h(N,s)$$

- We also calculate the Black-Scholes (x^{BS}) and Jennergren and Näslund (x^{JN}) ESO value and evaluate by Monte Carlo the corresponding expected squared hedging error implied by the risk-neutral delta hedging principles

Mean-Variance Optimal Hedging Frontier



Mean-Variance Optimal and Risk-Neutral Delta Hedging

We further note:

- **The expected hedging error is 0 for both x^{min} used with mean-variance optimal hedging, and x^{JN} used with risk-neutral delta hedging**
- **If the stock drift μ equals the discount rate r, then $x^{min} = x^{JN}$, and the mean-variance optimal and risk-neutral delta hedging are the same (analytical result)**

Results and Implications

Conclusions from a numerical study:

1. The expected squared hedging error is not negligible even for small λ. Therefore, the liquidation risk (diversification of Poisson jumps, market completeness) should be considered carefully when valuing ESOs
2. One can replicate an ESO less costly and with a smaller variance of the replication error than the benchmark JN model (if $\mu \neq r$)
3. Risk-neutral delta hedging is more risky (in terms of the squared replication error) than the suggested mean-variance optimal hedging (if $\mu \neq r$)

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