On Calibration of Stochastic and Fractional Stochastic Volatility Models

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Heston model

We consider the risk-neutral stock price model

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{v_t} S_t d\widetilde{W}_t^S, \\
dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} d\widetilde{W}_t^v, \\
d\widetilde{W}_t^S d\widetilde{W}_t^v &= \rho\, dt,
\end{aligned}
$$

with initial conditions $S_0 \geq 0$ and $v_0 \geq 0$, where

- S_t is the price of the underlying asset at time t,
- v_t is the instantaneous variance at time t,
- r is the risk-free rate,
- θ is the long run average price variance,
- κ is the rate at which v_t reverts to θ and
- σ is the volatility of the volatility.
- $(\widetilde{W}^S, \widetilde{W}^v)$ is a two-dimensional Wiener process under the risk-neutral measure $\widetilde{\mathbb{P}}$ with instantaneous correl. ρ.

S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *Review of Financial Studies*, vol. 6, no. 2, pp. 327–343, 1993.

European call option price $C(S, v, t)$ can be expressed as:

$$C(S, v, t) = S - Ke^{-r\tau}\frac{1}{\pi}\int_{0+i/2}^{\infty+i/2} e^{-ikX}\frac{\hat{H}(k, v, \tau)}{k^2 - ik}dk, \text{ where}$$

$$\hat{H}(k, v, \tau) = \exp\left(\frac{2\kappa\theta}{\sigma^2}\left[tg - \ln\left(\frac{1 - he^{-\xi t}}{1 - h}\right) + vg\left(\frac{1 - e^{-\xi t}}{1 - he^{-\xi t}}\right)\right]\right),$$

$$X = \ln(S/K) + r\tau$$

$$g = \frac{b - \xi}{2}, \quad h = \frac{b - \xi}{b + \xi}, \quad t = \frac{\sigma^2\tau}{2},$$

$$\xi = \sqrt{b^2 + \frac{4(k^2 - ik)}{\sigma^2}},$$

$$b = \frac{2}{\sigma^2}\left(ik\rho\sigma + \kappa\right).$$

A. L. Lewis, *Option valuation under stochastic volatility, with Mathematica code*. Finance Press, Newport Beach, CA, 2000.

Optimization problem, nonlinear least squares:

$$\inf_{\Theta} G(\Theta), \quad G(\Theta) = \sum_{i=1}^{N} w_i |C_i^{\Theta}(t, S_t, T_i, K_i) - C_i^*(T_i, K_i)|^2,$$

where

- N denotes the number of observed option prices,
- w_i is a weight,
- $C_i^*(T_i, K_i)$ is the market price of the call option observed at time t,
- C^{Θ} denotes the model price computed using vector of model parameters.

For Heston SV model we have $\Theta = (\kappa, \theta, \sigma, v_0, \rho)$.

Considered algorithms and their implementations

We tested

- global optimizers:
  in MATLAB's Global Optimization Toolbox:
  - genetic algorithm (GA) - function `ga()`
  - simulated annealing (SA) - function `simulannealbnd()`

  from `inberg.com`:
  - adaptive simulated annealing (ASA)
- local search method (LSQ):
  in MATLAB's Optimization Toolbox: function `lsqnonlin()`,
  - Gauss-Newton trust region,
  - Levenberg-Marquardt,

  in Microsoft Excel's solver
  - Generalized Reduced Gradient method,
- combination of both approaches, see later.

Maximum absolute relative error

$$\mathrm{MARE}(\Theta) = \max_i \frac{|C_i^\Theta - C_i^*|}{C_i^*}$$

and average of the absolute relative error

$$\mathrm{AARE}(\Theta) = \frac{1}{N}\sum_{i=1}^{N} \frac{|C_i^\Theta - C_i^*|}{C_i^*}$$

for $i = 1, \ldots, N$. Let $\delta_i > 0$ denote the bid ask spread.
We consider the following weights

$$\text{weight A:} \quad w_i = \frac{1}{|\delta_i|},$$

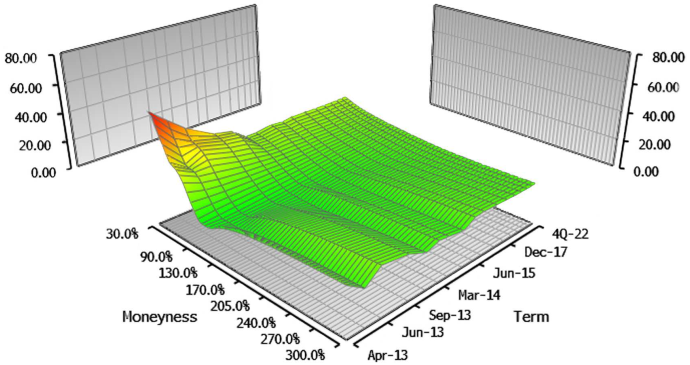
$$\text{weight B:} \quad w_i = \frac{1}{\delta_i^2},$$

$$\text{weight C:} \quad w_i = \frac{1}{\sqrt{\delta_i}}.$$

Empirical results for Heston model on real market data

DATA:

- Market prices obtained on March 19, 2013 from Bloomberg's Option Monitor for ODAX call options.
- We used a set of 107 options for 6 maturities.
- Volatility smile and term structure for DAX call options (sourced from Bloomberg Finance L.P.):

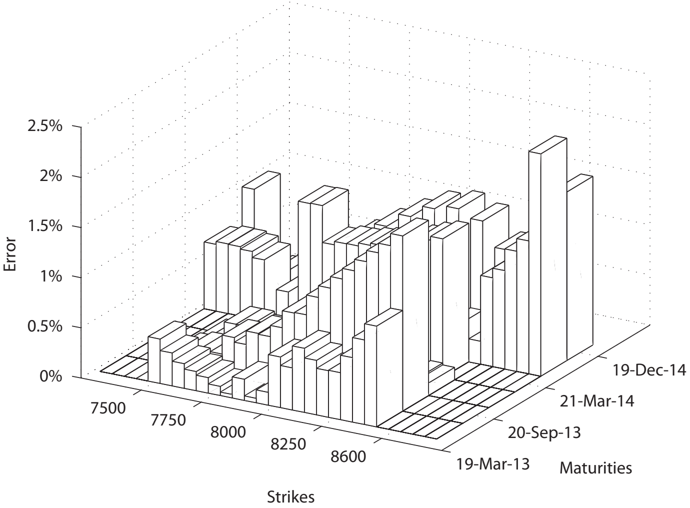


Calibration results

Algorithm	Weight	AARE	MARE	v_0	κ	θ	σ	ρ
GA	A	1.25%	12.46%	0.02897	0.68921	0.10313	0.79492	-0.53769
GA	B	2.10%	13.80%	0.03073	0.06405	0.94533	0.91248	-0.53915
GA	C	1.70%	18.35%	0.03300	0.83930	0.10826	1.14674	-0.49923
ASA	A	2.26%	19.51%	0.03876	0.80811	0.13781	1.63697	-0.46680
ASA	B	2.62%	28.65%	0.03721	1.45765	0.09663	1.86941	-0.37053
ASA	C	1.73%	19.82%	0.03550	1.22482	0.09508	1.44249	-0.49063
LSQ*	B	**0.58%**	3.10%	0.02382	1.75680	0.04953	0.42134	-0.84493
GA+Excel	A	1.25%	12.46%	0.02897	0.68922	0.10314	0.79490	-0.53769
GA+Excel	B	1.25%	12.46%	0.02896	0.68921	0.10314	0.79492	-0.53769
GA+Excel	C	1.25%	12.66%	0.02903	0.68932	0.10294	0.79464	-0.53763
ASA+Excel	A	1.73%	19.82%	0.03550	1.22482	0.09509	1.44248	-0.49062
ASA+Excel	B	1.78%	18.18%	0.03439	1.22399	0.09740	1.43711	-0.49115
ASA+Excel	C	1.73%	19.82%	0.03550	1.22482	0.09509	1.44248	-0.49062
GA+LSQ	A	0.67%	3.07%	0.02491	0.82270	0.07597	0.48665	-0.67099
GA+LSQ	B	0.65%	**2.22%**	0.02497	1.22136	0.06442	0.55993	-0.66255
GA+LSQ	C	0.68%	3.66%	0.02486	0.75195	0.07886	0.46936	-0.67266
ASA+LSQ	A	1.73%	19.82%	0.03550	1.22482	0.09508	1.44249	-0.49063
ASA+LSQ	B	1.71%	19.48%	0.03511	1.22672	0.09636	1.44194	-0.49089
ASA+LSQ	C	1.73%	19.82%	0.03550	1.22482	0.09508	1.44249	-0.49063

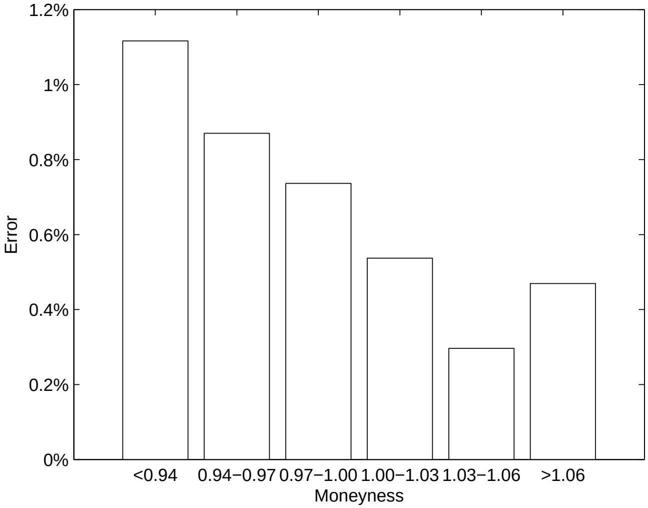
* initial guesses obtained by deterministic grid;

Results for pair GA and LSQ in terms of absolute relative errors:



Calibration results - GA+LSQ

Results for pair GA and LSQ in terms of absolute relative errors:



We consider the risk-neutral stock price model with approximative fractional stochastic volatility (FSV)

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^S + Y_t S_{t-} dN_t,$$
$$dv_t = -\kappa(v_t - \bar{v})dt + \xi v_t dB_t^H,$$

where

κ is a mean-reversion rate,
$\bar{v}$ stands for an average volatility level,
ξ is so-called volatility of volatility,
$(N_t)_{t\geq 0}$ is a Poisson process,
Y_t denotes an amplitude of a jump at t,
$(W_t^S)_{t\geq 0}$ ia s standard Wiener process,
$(B_t^H)_{t\geq 0}$ is an approximative fractional process.

A. Intarasit and P. Sattayatham, "An approximate formula of European option for fractional stochastic volatility jump-diffusion model," *Journal of Mathematics and Statistics*, vol. 7, no. 3, pp. 230–238, 2011.

Approximative fractional process

Let

$$B_t^H = \int_0^t (t - s + \varepsilon)^{H-1/2} dW_s,$$

where

H is a long-memory Hurst parameter in general $H \in [0, 1]$,

ε is a non-negative approximation factor,

$(W_t)_{t \geq 0}$ represents a standard Wiener process.

Long-range dependence of volatility if $H \in (0.5, 1]$.

If $\varepsilon > 0$ then B_t^H is a semi-martingale.

European call option price $V(\tau, K)$ can be expressed as:

$$V(\tau, K) = e^{x_t} P_1(x_t, v_t, \tau) - e^{-r\tau} K P_2(x_t, v_t, \tau),$$

where for $n = 1, 2$

$$P_n = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{i\phi \ln(K)} f_n}{i\phi} \right] d\phi,$$

$$f_n = \exp \left\{ C_n(\tau, \phi) + D_n(\tau, \phi) v_0 + i\phi \ln(S_t) + \psi(\phi)\tau \right\},$$

$$C_n(\tau, \phi) = r\phi i\tau + \theta Y_n \tau - \frac{2\theta}{\beta^2} \ln \left(\frac{1 - g_n e^{d_n \tau}}{1 - g_n} \right),$$

$$D_n(\tau, \phi) = Y_n \left(\frac{1 - e^{d_n \tau}}{1 - g_n e^{d_n \tau}} \right),$$

where all the unexplained terms follow...

Semi-closed form solution of the FSV model

For $n = 1, 2$

$$\psi = -\lambda i\phi \left(e^{\alpha_J + \gamma_J^2/2} - 1\right) + \lambda \left(e^{i\phi\alpha_J - \phi^2\gamma_J^2/2} - 1\right)$$

$$Y_n = \frac{b_n - \rho\beta\phi i + d_n}{\beta^2}$$

$$g_n = \frac{b_n - \rho\beta\phi i + d_n}{b_n - \rho\beta\phi i - d_n},$$

$$d_n = \sqrt{(\rho\beta\phi i - b_n)^2 - \beta^2(2u_n\phi i - \phi^2)},$$

$$\beta = \xi\varepsilon^{H-1/2}\sqrt{v_t}, \; u_1 = 1/2, \quad u_2 = -1/2, \; \theta = \kappa\bar{v},$$

$$b_1 = \kappa - (H - 1/2)\xi\varphi_t - \rho\beta,$$

$$b_2 = \kappa - (H - 1/2)\xi\varphi_t.$$

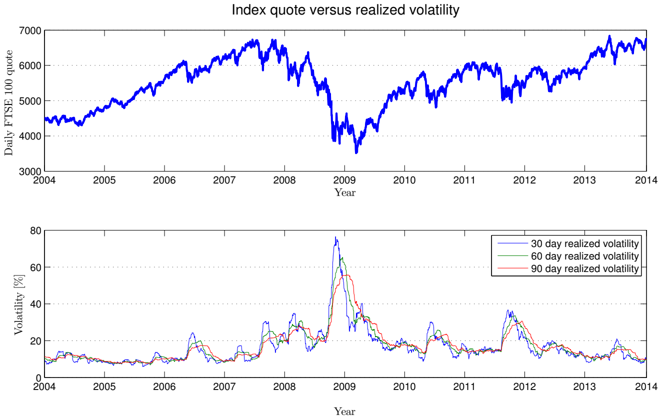
Rather complicated formula, but still 'Heston-like'.

The vector of parameters to be optimized will be $\Theta = (v_0, \kappa, \bar{v}, \xi, \rho, \lambda, \alpha_J, \gamma_J, H)$, where

v_0 initial volatility	κ mean reversion rate	$\bar{v}$ average volatility
ξ volatility of volatility	ρ correlation coef.	λ Poisson hazard rate
α_J expected jump size	γ_J variance of jump sizes	H Hurst parameter

DATA:

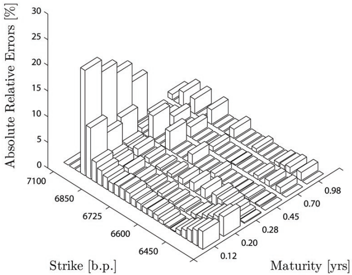
- Market prices obtained on January 8, 2014 from Bloomberg's Option Monitor for British FTSE 100 stock index call options.
- We used a set of 82 options for 6 maturities.



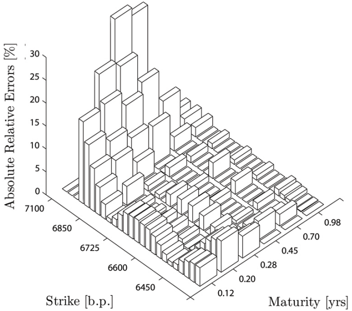
Model	Weights	Algorithm	AARE [%]	MARE [%]
FSV model	A	GA+LSQ	2.34	20.53
		SA+LSQ	2.34	20.53
Heston model	A	GA+LSQ	3.36	19.01
		SA+LSQ	4.43	29.34
FSV model	B	GA+LSQ	2.33	20.49
		SA+LSQ	2.34	20.53
Heston model	B	GA+LSQ	5.07	32.36
		SA+LSQ	4.15	23.33
FSV model	C	GA+LSQ	2.34	20.53
		SA+LSQ	2.34	20.53
Heston model	C	GA+LSQ	3.35	18.85
		SA+LSQ	3.52	19.93

The best calibration result in terms of AARE.

Results for pair GA and LSQ in terms of absolute relative errors for weights B :



FSV model



Heston model

Heston model:

- optimization problem is non-convex and may contain many local minima,
- local search method without a good initial guess may fail to achieve satisfactory results,
- we set a fine deterministic grid for initial starting points,
- best result of a trust region minimizer for these points (AARE=0.58%, MARE=3.10%) is taken as a reference point for comparison of less heuristic and more efficient approaches,
- with GA+LSQ we were able to get close (AARE=0.65%, MARE=2.22%).

M. Mrázek and J. Pospíšil, "Calibration and simulation of Heston model," 2014. [Under Review]

Conclusion continued

FSV model:

- a new 'Heston-like' semi-closed formula,
- first empirical calibration results,
- in some aspects better results than with Heston model.

J. Pospíšil and T. Sobotka, "Market calibration under a long memory stochastic volatility model," 2014. [Under Review]

Further issues:

- optimization techniques:

- performance and accuracy improvements of Gauss-Newton trust-region methods,
- variable metric methods for nonlinear least squares,
- fine tuning the global optimizers.

- presented approaches:

- calibration results with respect to exotic derivatives,
- hedging under the FSV model,
- large-scale parallel calibration of the models.