Lessons Learned from Stochastic Volatility Models Calibration and Simulation

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Outline

We consider a general SVJD model which covers several kinds of stochastic volatility processes and also different types of jumps

$$\begin{aligned} dS_t &= (r - \lambda\beta)S_t dt + \sqrt{v_t} S_t dW_t^S + S_{t-} dQ_t, \\ dv_t &= p(v_t)dt + q(v_t)dW_t^v, \\ dW_t^S dW_t^v &= \rho\, dt, \end{aligned}$$

where $p, q \in C^\infty(0, \infty)$ are general coefficient functions, r is the interest rate, ρ is the correlation of Wiener processes W_t^S and W_t^v, parameters λ and β correspond to a specific jump process Q_t, see below.

F. Baustian, M. Mrázek, J. Pospíšil and T. Sobotka, *Unifying approach to several stochastic volatility models with jumps*, manuscript under review, 2015.

$$dS_t = (r - \lambda\beta)S_t dt + \sqrt{v_t} S_t dW_t^S + S_{t-} dQ_t,$$
$$dv_t = p(v_t)dt + q(v_t)dW_t^v.$$

model	$p(v)$	$q(v)$
Heston/Bates	$\kappa(\theta - v)$	$\sigma\sqrt{v}$
3/2 model*	$\omega v - \tilde{\theta} v^2$	$\xi v^{\frac{3}{2}}$
Geometric BM	αv	ξv
Fractional SVJD**	$(H - 1/2)\psi_t \sigma\sqrt{v} + \kappa(\theta - v)$	$\varepsilon^{H-1/2}\sigma\sqrt{v}$

$^*\tilde{\theta} = -\frac{1}{2}\xi^2 + (1-\gamma)\rho\xi + \sqrt{(\theta + \frac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2}$,
$^{**}\psi_t = \int_0^t (t - s + \varepsilon)^{H-3/2} dW_s^{\psi}$.

- The jump process Q_t is a compound Poisson process $Q_t = \sum_{i=1}^{N_t} Y_i$.
- $Y_1, Y_2, \ldots$ are pairwise independent random variables with identically distributed jump sizes $\beta = \mathbb{E}[Y_i]$ for all $i \in \mathbb{N}$,
- N_t is a standard Poisson process with intensity λ independent of the Y_i.

Jumps examples:

- log-normal, $\ln(1+Y_i) \sim \mathcal{N}(\mu_J, \sigma_J^2)$, $\beta = \exp\left\{\mu_J + \frac{1}{2}\sigma_J^2\right\} - 1$.

  D.S. Bates, *Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options*, Review of Financial Studies 9(1), 1996.

- log-uniform, $\ln(1+Y_i) \sim \mathcal{U}(a,b)$, $\beta = \frac{e^b - e^a}{b-a} - 1$.

  G. Yan and F.B. Hanson, *Option Pricing for a Stochastic-Volatility Jump-Diffusion Model with Log-Uniform Jump-Amplitude*, Proceedings of American Control Conference, 2006.

The problem of pricing an option in a model with jumps corresponds to a partial integro-differential equation (PIDE). After substituting $x = \ln S$ we get the PIDE for $f(x, v, t) = V(e^x, v, t)$

$$-f_t = -rf + (r - \lambda\beta - \frac{1}{2}v)f_x + \frac{1}{2}vf_{xx} + pf_v + \frac{1}{2}q^2 f_{vv} + \rho q\sqrt{v}f_{xv}$$
$$+ \lambda \int_{-\infty}^{\infty} [f(x + y, v, t) - f(x, v, t)]\, \varphi(y)dy.$$

F.B. Hanson, *Applied stochastic processes and control for jump-diffusion*, Advances in Science and Control, 2007.

We can either solve the PIDE numerically or for simple contracts analytically - we can apply the complex Fourier transform similarly to what Lewis did for models without jumps

A.L. Lewis, *Option valuation under stochastic volatility, with Mathematica code*, Finance Press, 2000.

We want to apply the complex Fourier transform (Lewis, 2000)

$$\mathcal{F}[f] = \hat{f}(k,v,t) = \int_{-\infty}^{\infty} e^{ikx} f(x,v,t)dx$$

with the inverse transform

$$f(x,v,t) = \frac{1}{2\pi}\int_{-\infty+ik_i}^{\infty+ik_i} e^{-ikx}\hat{f}(k,v,t)dk$$

where k_i is some real number such that the line $(-\infty + ik_i, \infty + ik_i)$ is in some strip of regularity.

$$-\hat{f}_t = [-r - ik(r-\lambda\beta)]\,\hat{f} - \frac{1}{2}v(k^2 - ik)\hat{f} + (p - ik\rho q\sqrt{v})\hat{f}_v + \frac{1}{2}q^2\hat{f}_{vv}$$
$$+ \lambda\mathcal{F}\left[\int_{-\infty}^{\infty}[f(x+y,v,t) - f(x,v,t)]\,\varphi(y)dy\right].$$

We have to derive the Fourier transform of the integral term

$$\mathcal{F}\left[\int_{-\infty}^{\infty}[f(x+y,v,t)-f(x,v,t)]\,\varphi(y)dy\right]=\hat{f}(k,v,t)(\hat{\varphi}(-k)-1).$$

We substitute $\tau = T - t$ and define $\hat{h}(k,v,t)$ by

$$\hat{h}(k,v,t)=\exp\left(-\left[-r-ik(r-\lambda\beta)+\lambda(\hat{\varphi}(-k)-1)\right]\tau\right)\hat{f}(k,v,\tau)$$

and obtain the following equation

$$\hat{h}_\tau=\frac{1}{2}q^2(v)\hat{h}_{vv}+\left[p(v)-ik\rho(v)q(v)\sqrt{v}\right]\hat{h}_v-\frac{k^2-ik}{2}v\hat{h}.$$

We denote with $\hat{H}$ the solution of the equation with initial value $\hat{H}(k,v,0)=1$ which is regular as a function of $k=k_r+ik_i$ within a strip $k_1<k_i<k_2$.

Our unifying formula for the European call price V has the form

$$V(S,v,\tau) = S - Ke^{-r\tau}\frac{1}{2\pi}\int_{-\infty+ik_i}^{\infty+ik_i} e^{-ik\tilde{X}} e^{\lambda(\hat{\varphi}(-k)-1)\tau}\frac{\hat{H}(k,v,\tau)}{k^2-ik}\,dk,$$

where $\tilde{X} = \ln(S/K) + (r-\lambda\beta)\tau$ and $\max(k_1,0) < k_i < \min(1,k_2)$.

Financial claim	Payoff transform $\hat{w}(k)$	k-plane restrictions
Call option	$-\frac{K^{ik+1}}{k^2-ik}$	$\operatorname{Im} k > 1$
Put option	$-\frac{K^{ik+1}}{k^2-ik}$	$\operatorname{Im} k < 0$
Bull Spread option	$\frac{K_2^{ik+1}-K_1^{ik+1}}{k^2-ik}$	$\operatorname{Im} k > 0$
Bear Spread option	$\frac{K_1^{ik+1}-K_2^{ik+1}}{k^2-ik}$	$\operatorname{Im} k < 0$
Butterfly Spread	$\frac{2K_2^{ik+1}-K_1^{ik+1}-K_3^{ik+1}}{k^2-ik}$	none

Let $B^{\varepsilon}_t = \int\limits_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW_s$ be the approximative fractional Brownian motion, $\varepsilon > 0$, $H > 0.5$ (for $H = 0.5$ it is the standard Brownian motion).

Then the volatility process in the approximative fractional SVJD model

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB^{\varepsilon}_t,$$

can be rewritten as

$$dv_t = \left[(H-\frac{1}{2})\psi_t\sigma\sqrt{v_t} + \kappa(\theta - v_t)\right] dt + \varepsilon^{H-\frac{1}{2}}\sigma\sqrt{v_t}dW^v_t,$$

where $\psi_t = \int_0^t (t-s+\varepsilon)^{H-\frac{3}{2}} dW^{\psi}_s$.

J. Pospíšil and T. Sobotka, *Market calibration under a long memory stochastic volatility model*, manuscript under review, 2015.

We get the solution with

$$V(S,v,\tau) = S - Ke^{-r\tau}\frac{1}{2\pi}\int_{-\infty+i/2}^{\infty+i/2} e^{-ikX}\frac{\hat{H}_f(k,v,\tau)}{k^2-ik}\phi(-k)dk,$$

with $\hat{H}_f(k,v,\tau) = \exp(C_f(k,\tau)+D_f(k,\tau)v)$ and

$$C_f(k,\tau) = \kappa\theta Y\tau - \frac{2\kappa\theta}{B^2}\ln\left(\frac{1-ge^{d\tau}}{1-g}\right),$$

$$D_f(k,\tau) = Y\frac{1-e^{d\tau}}{1-ge^{d\tau}},$$

$$Y = -\frac{k^2-ik}{b-d}, g = \frac{b+d}{b-d}, d = \sqrt{b^2+B^2(k^2-ik)},$$

$$b = \kappa + ik\rho B,$$

$$B = \varepsilon^{H-\frac{1}{2}}\sigma.$$

Lesson 1: Efficiency of pricing formulas

Computational efficiency of our solution for studied models:

- we compare computational time with respect to the original and newly proposed formulas,
- three pricing tasks - 100 European call options with different times to maturity and strike prices:
    1. 100 parameter sets - *market calibration with good initial guess*,
    2. 1000 parameter sets - *average calibration using local search method*,
    3. 10000 parameter sets - *calibration with global optimization procedure*.
- parameter sets are randomly generated in given parameter bounds.

Computation were made on a reference PC (2x Intel Xeon E5-2630 CPU and 12 GB RAM).

Lesson 1: Efficiency of pricing formulas

Example: Bates model

Pricing approach	Task	Time [sec]	Speed-up factor
Original	#1	38.01	-
	#2	407.16	-
	#3	3396.74	-
Newly proposed	#1	9.37	4.06×
	#2	80.98	5.03×
	#3	926.10	3.67×

D.S. Bates, *Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options*, Review of Financial Studies 9(1), 1996.

F. Baustian, M. Mrázek, J. Pospíšil and T. Sobotka, *Unifying approach to several stochastic volatility models with jumps*, manuscript under review, 2015.

Lesson 1: Efficiency of pricing formulas

Bates model

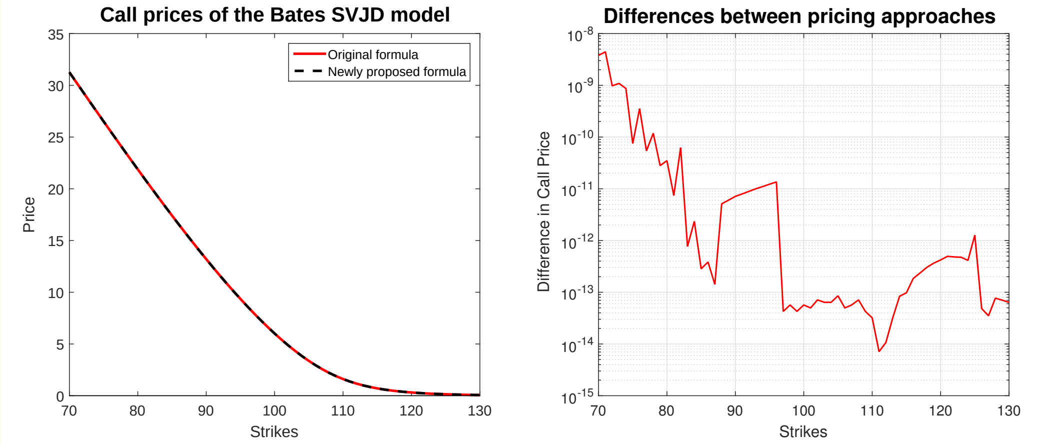


Figure: Parameters: $v_0 = 0.025$, $\kappa = 0.98$, $\theta = 0.07$, $\sigma = 0.54$, $\rho = -0.65$, $\lambda = 0.5$, $\mu_J = -0.05$, $\sigma_J = 0.1$ for $S_0 = 100$, $\tau = 0.5$, $r = 0.03$.

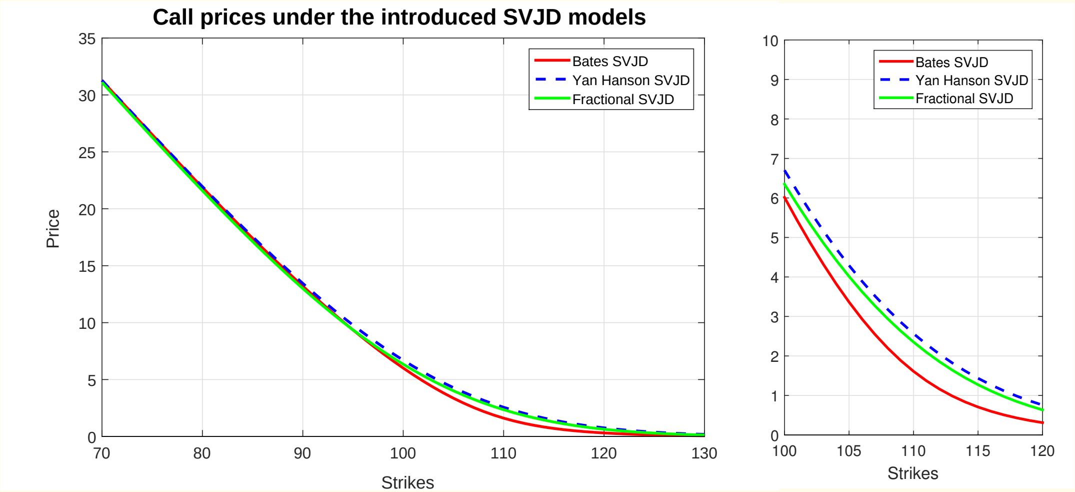


Figure: Option price as a function of the strike price for a call option with maturity 0.5 years and $S_0 = 100$, $r = 0.03$, $H = 0.7$.

Lesson 2: Pitfalls of numerical integration

For some model parameters, we can observe (especially for adaptive quadrature algorithms):

- an enormous increase in function evaluations,
- serious precision problems as well as
- a significant increase in computational time.

Problems are caused by inaccurately evaluated integrands:

- all models (including Heston) affected,
- especially sensitive to the value of volatility of volatility σ,
- standard double vs. variable precision arithmetic (vpa).

J. Daněk and J. Pospíšil, *Numerical integration of inaccurately evaluated functions*. In Technical Computing Prague 2015. Prague: Czech Technical University, 2015. p. 1-11.

J. Daněk and J. Pospíšil, *Numerical aspects of integration in semi-closed option pricing formulas under jump-diffusion stochastic volatility models*, manuscript in preparation.

Lesson 2: Pitfalls of numerical integration

FSV Example: Global view to integrated function

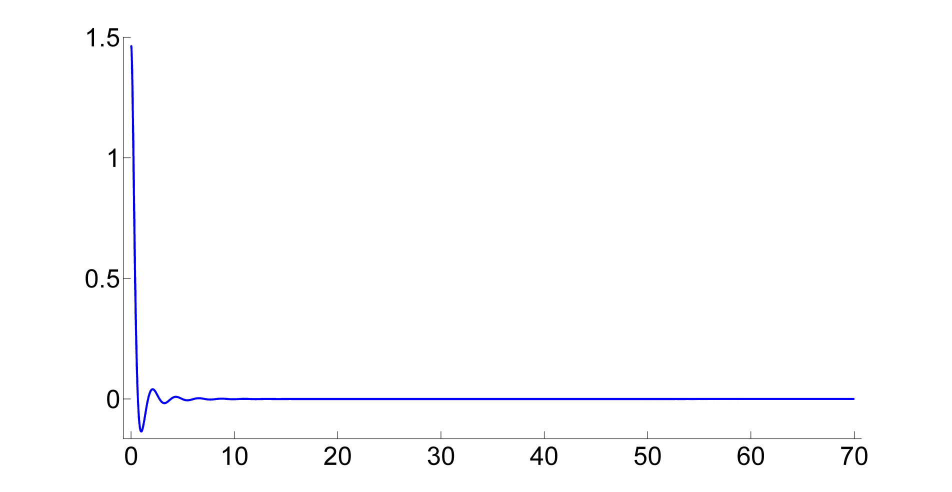


Figure: Parameters: $v_0 = 0.1$, $\kappa = 2.1$, $\theta = 0.4$, $\sigma = 0.002$, $\rho = -0.3$, $\lambda = 25$, $\mu_J = -4$, $\sigma_J = 1.7$, $H = 0.8$, for $S_0 = 6721.8$, $K = 6250$, $\tau = 0.120548$, $r = 0.009$.

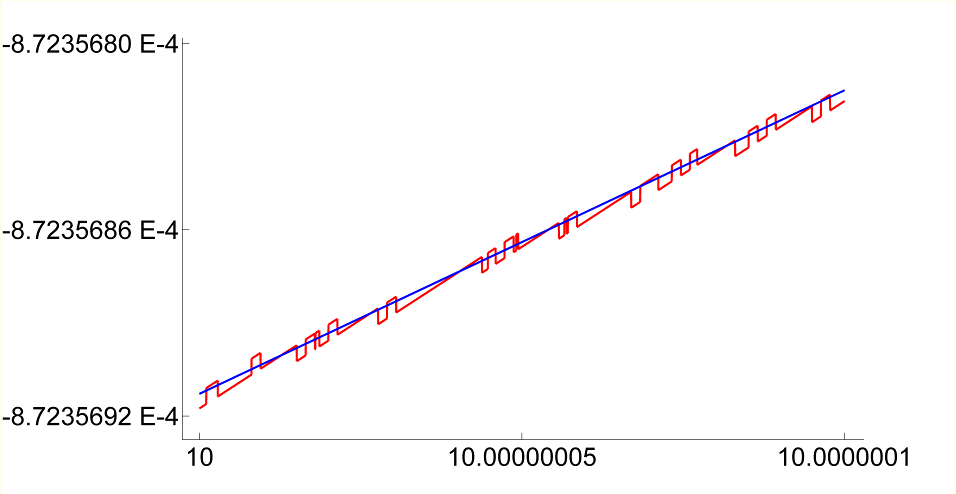


Figure: Same parameters as before. Inaccurately enumerated values in standard double precision (red) and in vpa (blue) evaluated with 40 significant digits.

Optimization problem, nonlinear least squares:

$$\inf_{\Theta} G(\Theta), \quad G(\Theta) = \sum_{i=1}^{N} w_i |C_i^{\Theta}(t, S_t, T_i, K_i) - C_i^*(T_i, K_i)|^2,$$

where

N	denotes the number of observed option prices,
w_i	is a weight,
$C_i^*(T_i, K_i)$	is the market price of the call option observed at time t,
C^{Θ}	denotes the model price computed using vector of model parameters.

Heston model: $\Theta = (v_0, \kappa, \theta, \sigma, \rho)$,
Bates model: $\Theta = (v_0, \kappa, \theta, \sigma, \rho, \lambda, \mu_J, \sigma_J)$,
Yan-Hanson model: $\Theta = (v_0, \kappa, \theta, \sigma, \rho, \lambda, a, b)$,
FSV model: $\Theta = (v_0, \kappa, \theta, \sigma, \rho, \lambda, \mu_J, \sigma_J, H)$.

- global optimizers:
  in MATLAB's Global Optimization Toolbox:
  - genetic algorithm (GA) - function `ga()`
  - simulated annealing (SA) - function `simulannealbnd()`

  from `inberg.com`:
  - adaptive simulated annealing (ASA)
- local search method (LSQ):
  in MATLAB's Optimization Toolbox: function `lsqnonlin()`,
  - Gauss-Newton trust region,
  - Levenberg-Marquardt,

  in Microsoft Excel's solver
  - Generalized Reduced Gradient method,
- combination of both approaches, see later.

Maximum and average of absolute relative error

$$\mathrm{MARE}(\Theta) = \max_{i=1,\dots,N} \frac{|C_i^\Theta - C_i^*|}{C_i^*}, \qquad \mathrm{AARE}(\Theta) = \frac{1}{N}\sum_{i=1}^{N} \frac{|C_i^\Theta - C_i^*|}{C_i^*}$$

Let $\delta_i > 0$ denote the bid ask spread.
We consider the following weights

$$w_i^A = \frac{|\delta_i|^{-1}}{\sum_{j=1}^{N} |\delta_j|^{-1}}, \quad w_i^B = \frac{\delta_i^{-2}}{\sum_{j=1}^{N} \delta_j^{-2}}, \quad w_i^C = \frac{\delta_i^{-1/2}}{\sum_{j=1}^{N} \delta_j^{-1/2}},$$

$$w_i^D = \frac{\mathrm{Vega}_i^2}{\sum_{j=1}^{N} \mathrm{Vega}_i^2}, \quad w_i^E = \frac{1}{N}.$$

Real market data

- option data really difficult to get for academic purposes,
- paid services such as Bloomberg Professional or Thomson Reuters Eikon rather expensive,
- exotic options almost impossible to get even with Bloomberg - only as OTC (private) contracts,
- a new cooperation with someone who can provide data welcomed.

We present an example:

- 97 ODAX calls traded on 18/03/2013 ranging from 86.5% to 112.0% moneyness across 5 maturities from ca 13.5 weeks to 1.76 years;
- 107 ODAX calls traded on 19/03/2013 ranging from 88.5% to 112.2% moneyness across 6 maturities from ca 13.4 weeks to 1.75 years.

Lesson 3: Calibration of SV models

Data source: Bloomberg Finance L.P.

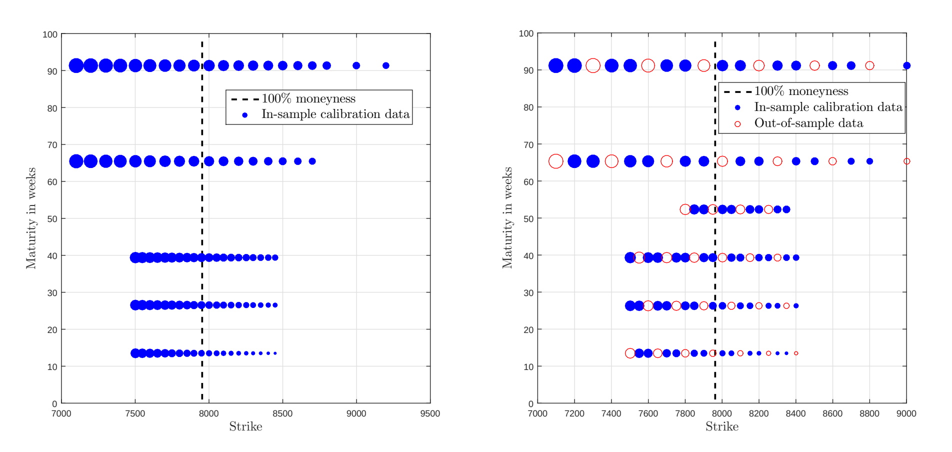


Figure: Option price structure in the strike/maturity plane for ODAX call 18/03/2013 on the left and for 19/03/2013 on the right resp. The center of each circle corresponds to the strike/maturity parameters of the traded contract, circle diameter is proportionate to the option premium.

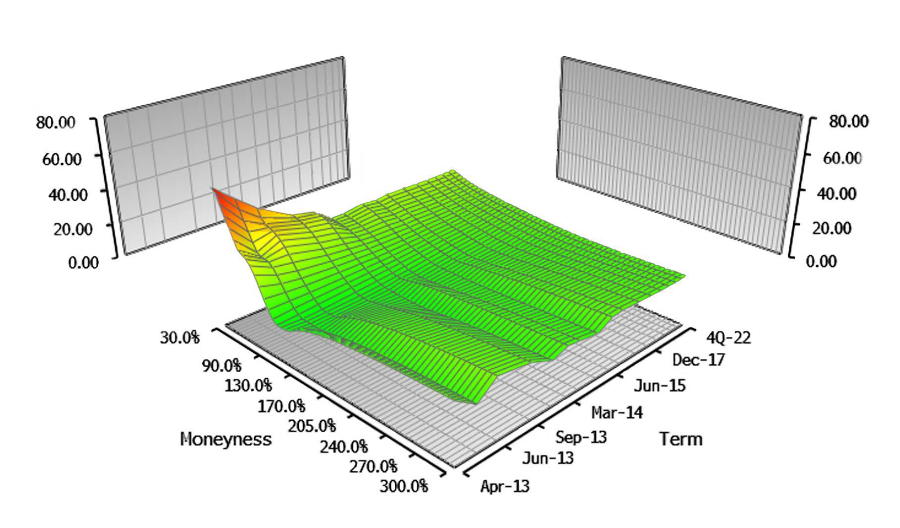


Figure: Volatility smile and term structure for ODAX calls 19/03/2013.

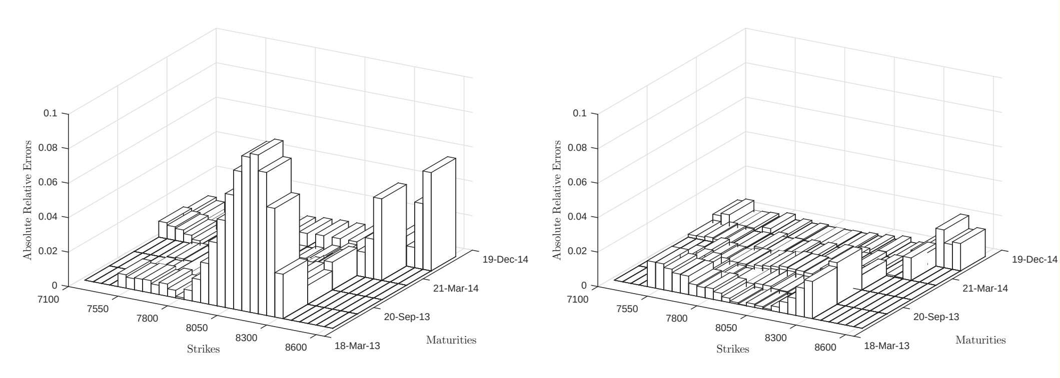


Figure: Calibration results for the FSV model using SA (left figure) and SA combined with LSQ.

Lesson 3: Calibration of SV models

Calibration results - GA+LSQ

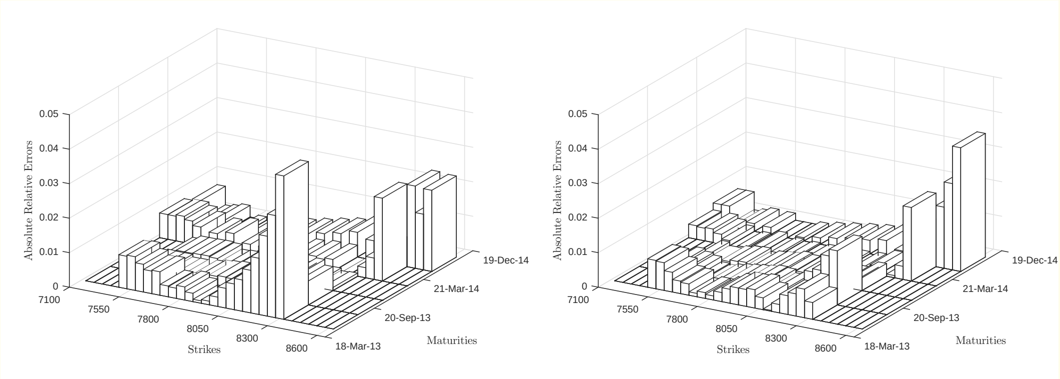


Figure: Results of calibration for pair GA and LSQ for weights C - Heston model on the left and FSV model on the right.

- Optimization problem is non-convex and may contain many local minima,
- local search method without a good initial guess may fail to achieve satisfactory results,
- we can set a fine deterministic grid for initial starting points (rather time consuming, even in parallel environment), or we can use several iterations of a global optimizer (e.g. suficiently large population in GA),
- Vega weights are least suitable.

M. Mrázek, J. Pospíšil and T. Sobotka, *On Optimization Techniques for Calibration of Stochastic Volatility Models*, In Applied Numerical Mathematics and Scientific Computation. Athens: Europment, 2014. pp. 34–40.

M. Mrázek, J. Pospíšil and T. Sobotka, *On calibration of stochastic and fractional stochastic volatility models*. European Journal of Operational Research, in press, 2016, doi:10.1016/j.ejor.2016.04.033.

Lesson 4: Robustness and sensitivity analysis

- New approach to robustness and sensitivity analysis for SV models is introduced,
- bootstrapping and Monte Carlo filtering techniques are applied,
- we address the impact of jumps and long memory in practice.

We present an example:

- European call options on AAPL. Four data sets from slightly different time periods: 01/04/2015, 15/04/2015, 01/05/2015 and 15/05/2015.
- Behavioural set of parameters - for which AARE is in lower 3/8 quantile, non-behavioural set - AARE in upper 3/8 quantile,

Table: Importance of λ for calibrations AAPL options on all four datasets - for all we were able to reject the null hypothesis (both sets from the same distribution) at significance level 5%.

Data sets	01/04/2015	15/04/2015	01/05/2015	15/05/2015
p-value	1.30%	0.43%	8.45e-12%	3.56%

J. Pospíšil, T. Sobotka and P. Ziegler, *Robustness and sensitivity analyses for stochastic volatility models under uncertain data structure*, 2015, manuscript in revision.

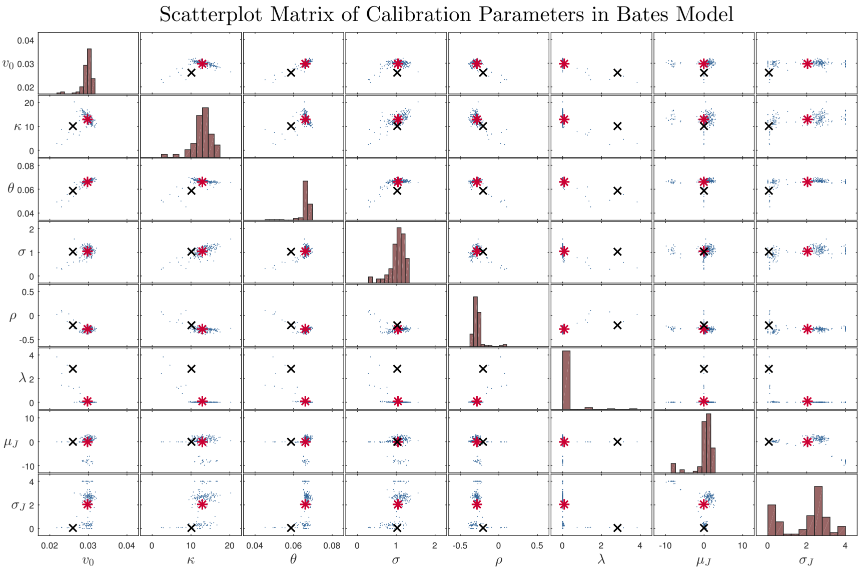
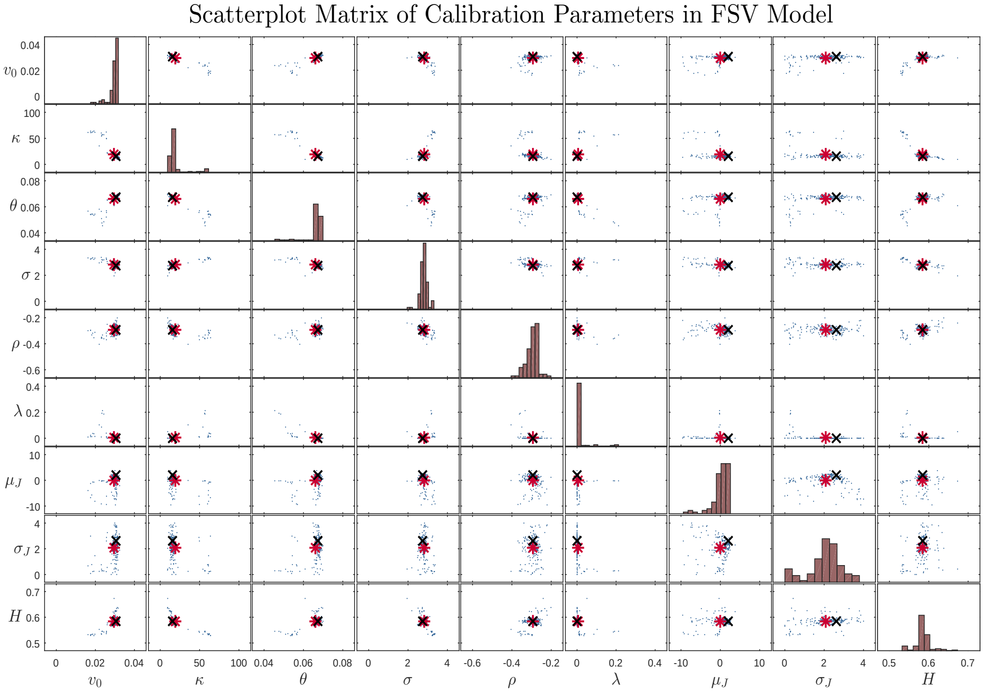


Figure: 15/05/2015. Diagonal elements depict histograms of parameter values obtained by bootstrap calibrations (e.g. the fist histogram corresponds to the values of v_0). Off-diagonal elements illustrate a dependence structure for each parameter pair. In those figures, a black cross represents the reference value of the specific parameter (calibration to all data) and by a red star we depict the bootstrap estimate of the value.



Scatterplot Matrix of Calibration Parameters in FSV Model
v_0
κ
θ
σ
ρ
λ
μ_J
σ_J
H

Lesson 5: Simulation of SV models

Simulation of the CIR volatility process

- Euler scheme, Milstein scheme, absorption or reflection technique for positivity,
- some schemes evidently wrong: Kahl-Jäckel (2006) scheme, Haastrecht and Pelsser (2010),
- exact scheme by Broadie and Kaya (2006) - problems with huge values of modiffied Bessel functions,
- QE scheme by Andersen (2008) - samples from approximated non-central chi-square distribution, probably the most efficient method.

For models with jumps a simple modification of the QE scheme is available.

For variance reduction we use antithetic variates method.

M. Mrázek and J. Pospíšil, *Calibration and simulation of Heston model*, 2015, manuscript in revision.

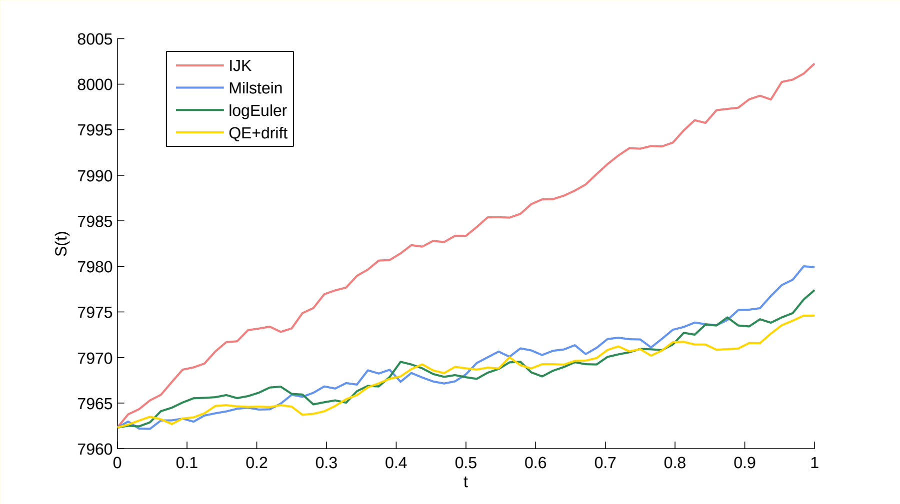


Figure: Mean of 100 000 paths. Parameters: $v_0 = 0.02497$, $\kappa = 1.22136$, $\theta = 0.06442$, $\sigma = 0.55993$, $\rho = -0.66255$, $S_0 = 7962.31$, $r = 0.00207$, $T = 1$. Time step $\Delta = 2^{-6}$.

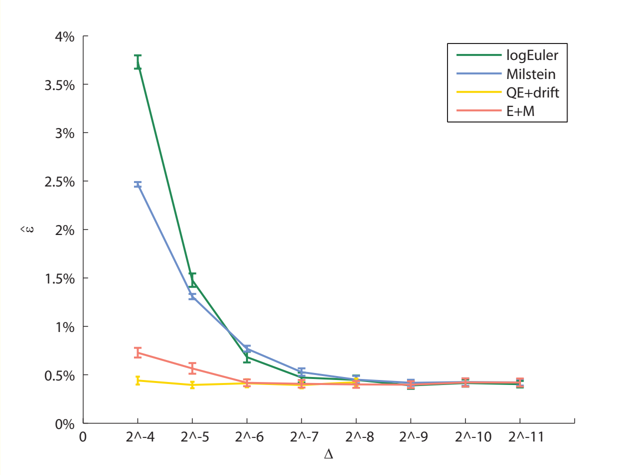


Figure: Convergence of schemes. 100 000 simulated paths, 100 batches. Parameters: $v_0 = 0.02497$, $\kappa = 1.22136$, $\theta = 0.06442$, $\sigma = 0.55993$, $\rho = -0.66255$, $S_0 = 7962.31$, $r = 0.00207$, $T = 1$. Time step $\Delta = 2^{-4}, \ldots, 2^{-11}$.

Conclusion and further issues

FSV model:

- a new semi-closed formula,
- first empirical calibration results,
- in some aspects better results than with Heston model.

Further issues:

- performance and accuracy improvements of Gauss-Newton trust-region methods,
- variable metric methods for nonlinear least squares,
- fine tuning the global optimizers.
- efficient pricing of exotic derivatives,
- hedging under the FSV model,
- large-scale parallel calibration of the models.

Conclusion and further issues

- We are analyzing the PIDE that determines the Fundamental solution. What assumptions do we have to make on p and q to get:
  - existence and uniqueness of the solution or even an explicit solution,
  - certain degree of regulartity for the solution, especially with respect to k on the line $k = k_r + ik_i$ with $k_r \in \mathbb{R}$ and fixed $k_i \in \mathbb{R}$.
- We are studying the numerical integration techniques of the pricing formulas. There are many nontrivial open issues involving both the accuracy and the speed of calculation, some of them can be solved using the vpa only.
- For non-European type of contracts we are studying a numerical solution of the corresponding PIDE using finite difference methods and finite element methods.
- We are studying models with rough volatility, i.e. with driving process being not only approximative fractional, but for example standard fractional Brownian motion.

Thank you for your attention!