Fitting NURBS to Financial Curves

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Outline

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- Examples of NURBS basis functions

Fitting NURBS to functions
- NURBS fit to a smooth function
- NURBS fit to a non-smooth but continous function
- NURBS fit to a discontinuous function

Finite element method with NURBS basis functions

Conclusion and further issues

A knot vector $C = (c_1, \ldots, c_m)^T$ is a nondecreasing vector of m real-valued coordinates in the parameter space such that

$$m = n + p + 1,$$

where n is the number of basis functions used to construct the B-spline curve and p is the polynomial order. Partitioning of the parameter space into elements can be either uniform or non-uniform. A knot vector is said to be open if its first and last knot values are repeated $p + 1$ times.

Example

$p = 3, m = 13, n = 9, \quad (0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6)^T$

$p = 3, m = 15, n = 9, \quad (0, 0, 0, 0, 1, 2, 3, 3, 3, 4, 5, 6, 6, 6, 6)^T$ - knot 3 repeated 3 times

The B-spline basis of the degree zero ($p = 0$) is defined as a piecewise constant

$$N_{i,0}(\xi) = \begin{cases} 1, & c_i \leq \xi < c_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 1, \ldots, n$. The higher order B-spline basis functions are defined recursively as

$$N_{i,p}(\xi) = \frac{\xi - c_i}{c_{i+p} - c_i} N_{i,p-1}(\xi) + \frac{c_{i+p+1} - \xi}{c_{i+p+1} - c_{i+1}} N_{i+1,p-1}(\xi),$$

for $i = 1, 2, \ldots, n$ and $p = 1, 2, 3, \ldots$. Some properties:

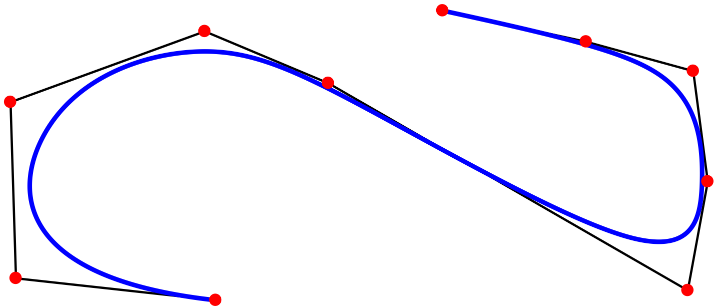
1. the basis functions $N_{i,p}(\xi)$ are all piecewise polynomial,
2. the sum of all basis functions $\sum_{i=1}^{n} N_{i,p}(\xi)$ for $\xi \in [c_1, c_m]$ is equal to a function being identically equal to one,
3. all basis functions are non-negative, i.e. $N_{i,p}(\xi) \geq 0$ for all ξ.

A B-spline curve in $\mathbb{R}^d$ is defined as a linear combination of B-spline basis functions, the vector-valued coefficients are referred to as control points. Given n basis functions $N_{i,p}(\xi), i = 1, 2, \ldots, n$ and corresponding control points $\mathbf{P}_i \in \mathbb{R}^d, i = 1, 2, \ldots, n$, a piecewise polynomial B-spline curve of order p is given by

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{P}_i. \tag{1}$$

Example

2D cubic curve



Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ be a weight vector such that $w_i > 0$ for $i = 1, 2, \ldots, n$. Then, we define NURBS basis functions by

$$R_i^p(\xi) = \frac{w_i N_{i,p}(\xi)}{\sum_{j=1}^{n} w_j N_{j,p}(\xi)}.$$

Some properties

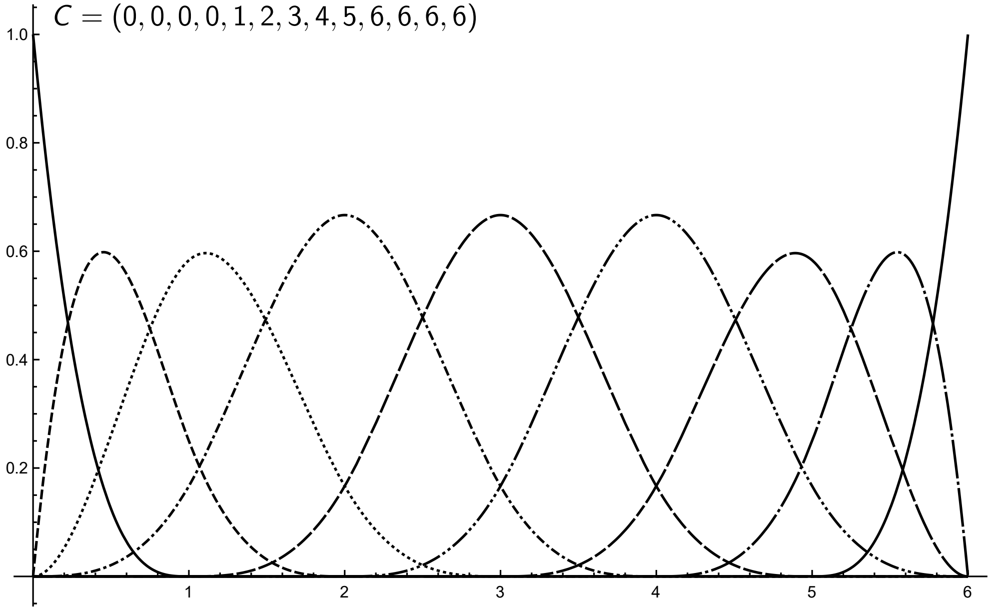
1. the basis functions $R_i^p(\xi)$ are piecewise rational; since it is defined as a ratio of two piecewise polynomials of order p, it is also often referred to as having order p and common names quadratic $(p = 2)$ basis function, cubic $(p = 3)$ and similar are often used in this sense,
2. the sum of all basis functions $\sum_{i=1}^{n} R_i^p(\xi)$ is equal to a function being identically one,
3. all the basis functions $R_i^p(\xi)$ are nonnegative,
4. every basis function $R_i^p(\xi)$ has the same support as the corresponding $N_{i,p}(\xi)$.

A NURBS curve is defined similarly as the B-spline curve

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_i^p(\xi)\mathbf{P}_i.$$

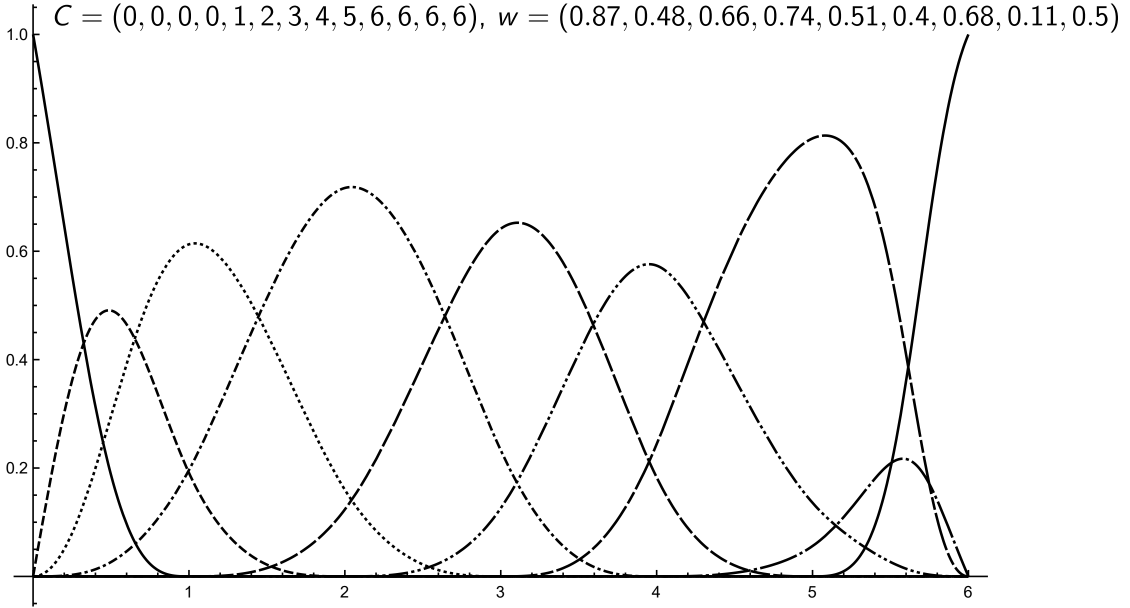
Examples of NURBS basis functions

A cubic ($p = 3$) B-spline basis with open, uniform knot vector



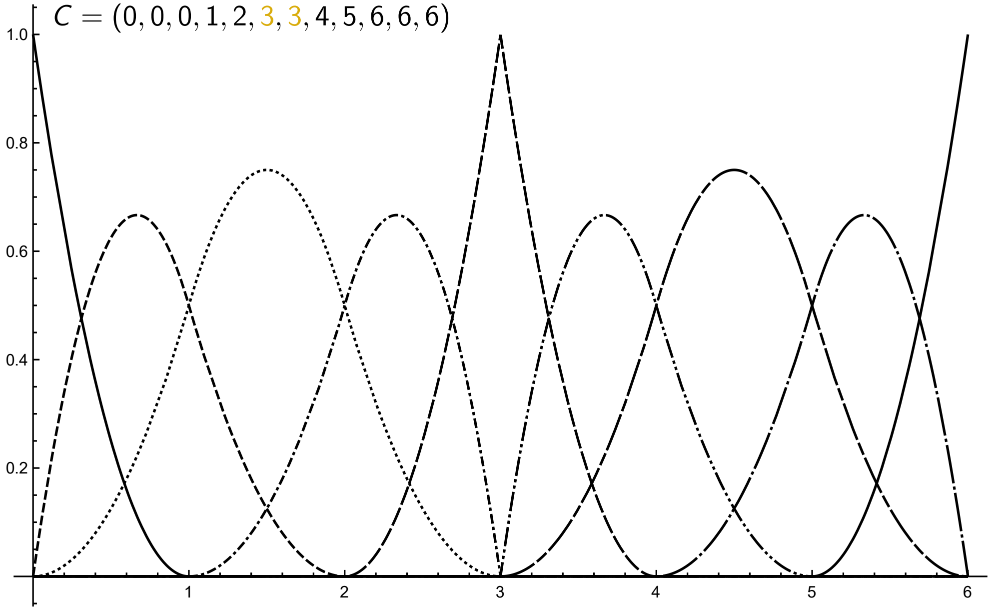
Examples of NURBS basis functions

A cubic ($p = 3$) NURBS basis with open, uniform knot vector



Examples of NURBS basis functions

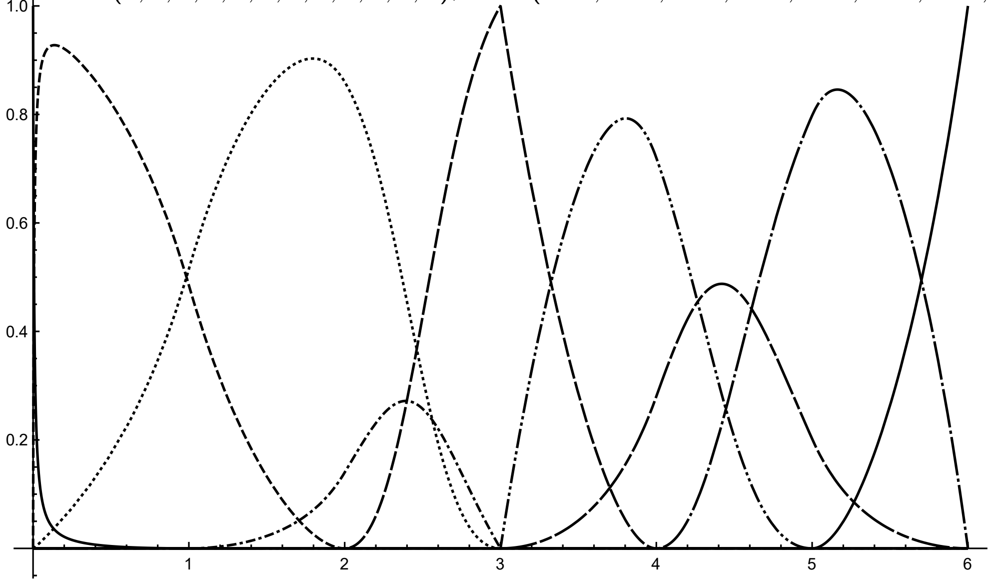
A quadratic ($p = 2$) B-spline basis with open, non-uniform knot vector



Examples of NURBS basis functions

A quadratic ($p = 2$) NURBS basis with open, non-uniform knot vector

$$C = (0, 0, 0, 1, 2, 3, 3, 4, 5, 6, 6, 6), \; w = (0.01, 0.81, 0.86, 0.14, 0.58, 0.54, 0.21, 0.83, 0.78)$$



Fitting NURBS to functions

Given a knot vector C and the order p, we are looking for the best fit to the function $f(\xi)$, $\xi \in [c_1, c_m]$ either by a B-spline – a linear combination of $n = m - p - 1$ B-spline basis functions $N_{i,p}(\xi)$,

$$\hat{f}_{\mathrm{bs}}(\xi) = \sum_{i=1}^{n} \hat{a}_i N_{i,p}(\xi),$$

where coefficients $\hat{a}_i$ are to be determined, or by a NURBS

$$\hat{f}_{\mathrm{nrb}}(\xi) = \sum_{i=1}^{n} \hat{b}_i R_i^p(\xi),$$

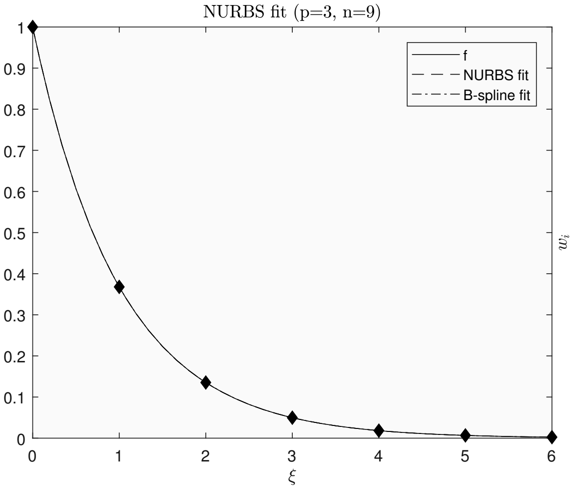
where $\hat{b}_i$ and the weights $w_i > 0$, $i = 1, 2, \ldots, n$ are also to be determined. The best fit will be measured by the mean L^2 errors

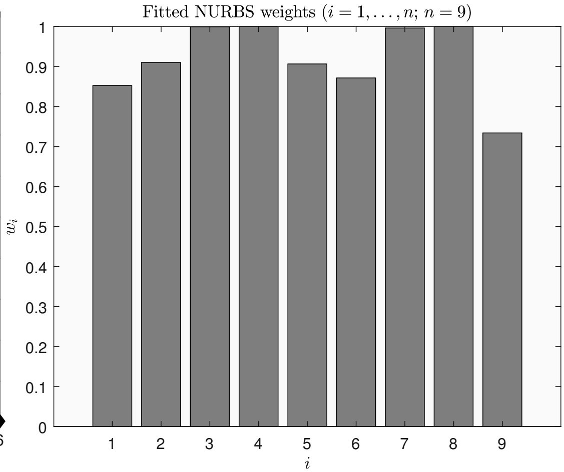
$$\hat{\epsilon}_{\mathrm{bs}} = \frac{1}{c_m - c_1} \int_{c_1}^{c_m} (f(\xi) - \hat{f}_{\mathrm{bs}}(\xi))^2 \,\mathrm{d}\xi \quad \text{and} \quad \hat{\epsilon}_{\mathrm{nrb}} = \frac{1}{c_m - c_1} \int_{c_1}^{c_m} (f(\xi) - \hat{f}_{\mathrm{nrb}}(\xi))^2 \,\mathrm{d}\xi,$$

where $\hat{\epsilon}_{\mathrm{nrb}}$ is minimized with respect to the weights w_i.

NURBS fit to a smooth function

discount curve $f(\xi) = \mathrm{e}^{-\xi}$, $\xi \in [0, 6]$

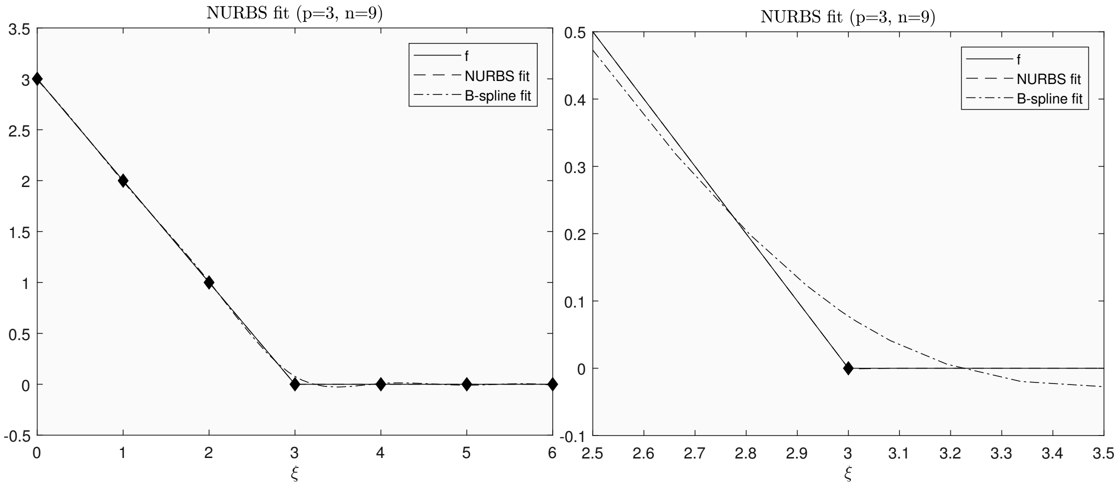




$p = 3,\ m = 13,\ n = 9,\ \hat{\epsilon}_{\mathrm{bs}} \doteq \texttt{2.7751e-05}$ and $\hat{\epsilon}_{\mathrm{nrb}} \doteq \texttt{5.8463e-07}$

NURBS fit to a non-smooth but continous function

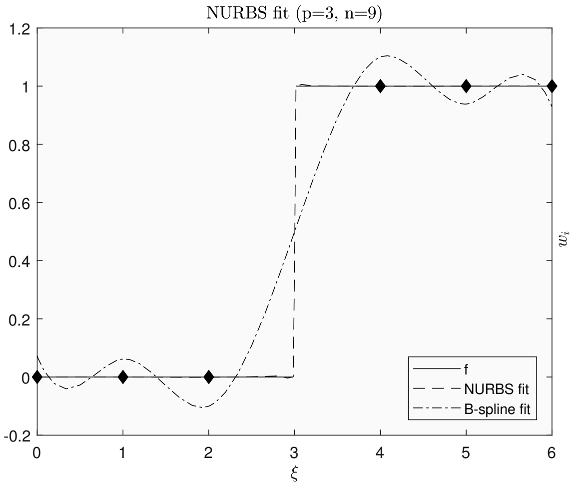
put option payoff $f(\xi) = \max(3 - \xi, 0)$, $\xi \in [0, 6]$

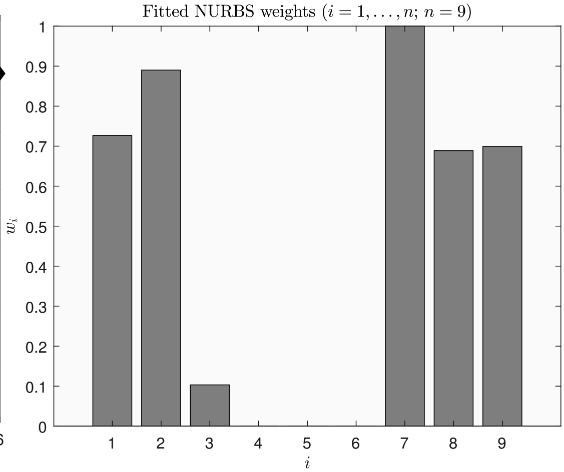


$p = 3$, $m = 13$, $n = 9$, $\hat{\epsilon}_{\text{bs}} \doteq 6.3688\text{e-}03$ and $\hat{\epsilon}_{\text{nrb}} \doteq 2.8930\text{e-}05$

NURBS fit to a discontinuous function

digital option payoff $f(\xi) = 1$ for $\xi \geq 3$ and zero otherwise

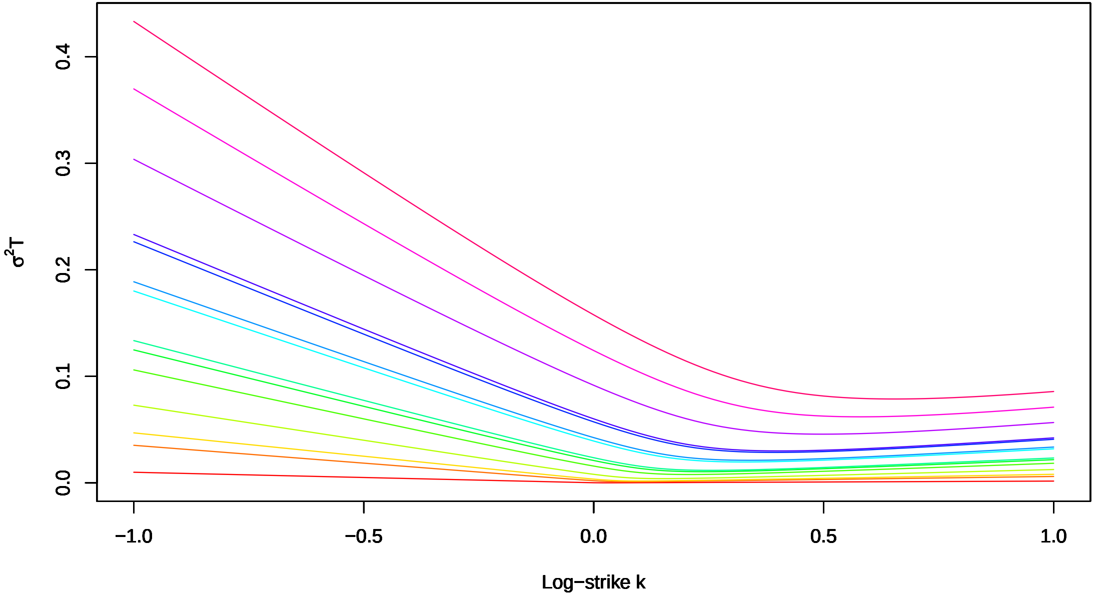




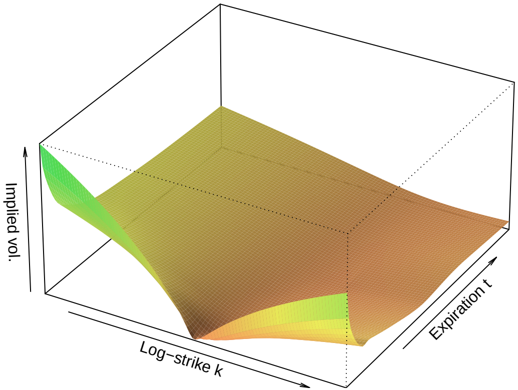
$p = 3$, $m = 13$, $n = 9$, $\hat{\epsilon}_{\text{bs}} \doteq$ 5.6854e-02 and $\hat{\epsilon}_{\text{nrb}} \doteq$ 4.7959e-04

Fitting NURBS to functions

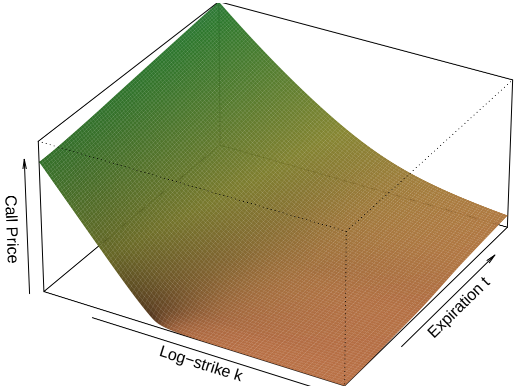
Typical implied volatility fits



Although the rectangular domains considered in option pricing equations are rather simple from the geometrical point of view, price surfaces obtained as a solution to these equations can be on the other hand quite complex.



implied volatility surface (smile)



price surface (hockey stick)

- Constant volatility models - fundamental works

F. S. Black and M. S. Scholes (1973). *The pricing of options and corporate liabilities*, J. Polit. Econ. 81(3), 637–654. doi: 10.1086/260062.

R. C. Merton (1976). *Option pricing when underlying stock returns are discontinuous*, J. Financ. Econ. 3(1–2), 125–144. doi: 10.1016/0304-405X(76)90022-2.

- Stochastic volatility models

S. L. Heston (1993). *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Rev. Financ. Stud. 6(2), 327–343 . doi: 10.1093/rfs/6.2.327.

D. S. Bates (1996). *Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options*, Rev. Financ. Stud. 9(1), 69–107. doi: 10.1093/rf s/9.1.69.

- Approximative fractional stochastic volatility jump diffusion (AFSVJD) model

J. Pospíšil and T. Sobotka (2016). *Market calibration under a long memory stochastic volatility model*, Appl. Math. Finance 23(5), 323–343. doi: 10.1080/1350486X.2017.1279977.

In Merton (1976) model we arrive at the problem of solving the localized PIDE for unknown function $f : [0, T] \times [0, \bar{s}] \to \mathbb{R}_0^+$

$$\begin{cases} f_\tau(\tau, s) - \frac{1}{2}\sigma^2 s^2 f_{ss}(\tau, s) - (r - \lambda\beta) s f_s(\tau, s) + r f(\tau, s) \\ \qquad -\lambda \int_0^{+\infty} [f(\tau, sy) - f(\tau, y)] \varphi(y)\, \mathrm{d}y = 0, \quad \tau \in (0, T), s \in (0, \bar{s}), \\ f(\tau, s) = h_D(\tau), \quad s \in \Gamma_D, \\ f_s(\tau, s) = h_N(\tau), \quad s \in \Gamma_N, \\ f(0, s) = \phi(s). \end{cases}$$

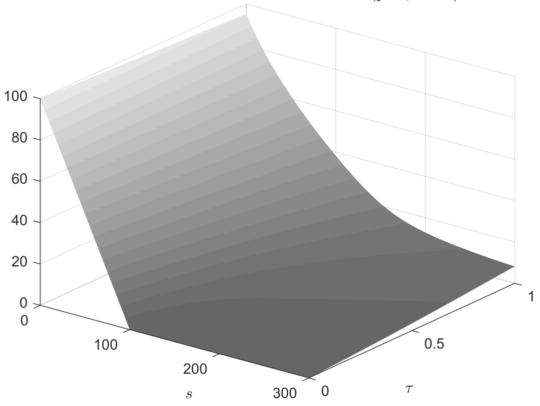
where $\emptyset \neq \Gamma_D \subset \{0, \bar{s}\}$ and $\Gamma_N \subset \{0, \bar{s}\}$. In the underlying stochastic model, jumps occur at Poisson distributed times with intensity λ and their log-sizes are normally distributed with mean μ_J and variance σ_J^2, we set

$$\beta = \exp\left\{\mu_J + \frac{1}{2}\sigma_J^2\right\} - 1 \quad \text{and} \quad \varphi(y) = \frac{1}{\sigma_J\sqrt{2\pi}} \exp\left\{-\frac{(y - \mu_J)^2}{2\sigma_J^2}\right\}.$$

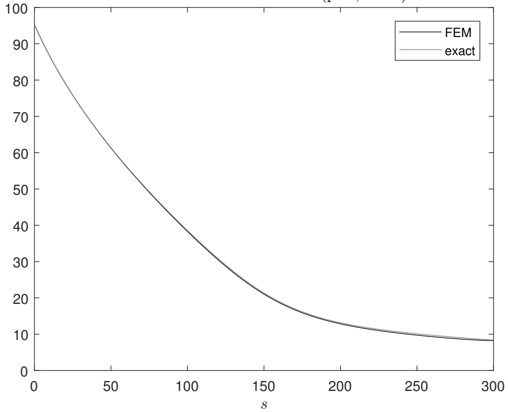
FEM with NURBS basis functions

Merton model - solution

Merton model: NURBS FEM solution (p=3, n=35)



Merton model: NURBS FEM solution (p=3, n=35) in $\tau = 1$



In AFSVJD (Pospíšil and Sobotka, 2016) model, the pricing PIDE for unknown function $f : [0, T] \times [0, \bar{s}] \times [0, \bar{v}] \to \mathbb{R}_0^+$

$$
\begin{cases}
f_\tau - \frac{1}{2} v\, s^2 f_{ss} - \rho q(v)\sqrt{v}\, s\, f_{sv} - \frac{1}{2} q^2(v) f_{vv} \\
\qquad -(r - \lambda\beta)\, s\, f_s - p(v) f_v + rf - \\
\qquad -\lambda \int_0^{+\infty} [f(\tau, sy) - f(\tau, y)]\, \varphi(y)\, \mathrm{d}y = 0, \quad \tau \in (0, T), s \in (0, \bar{s}), v \in (0, \bar{v}), \\
f(\tau, s, v) = h_D(\tau, s, v), \quad (s, v) \in \Gamma_D, \\
\nabla f(\tau, s, v) \cdot \overrightarrow{n_P} = h_N(\tau, s, v), \quad (s, v) \in \Gamma_N, \\
f(0, s, v) = \phi(s, v),
\end{cases}
$$

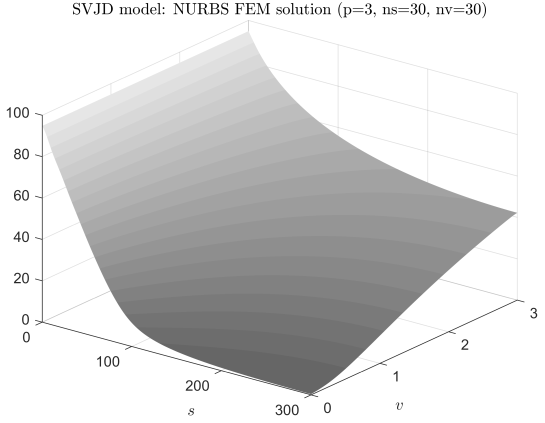
where $p(v) = \kappa(\theta - v), q(v) = \varepsilon^{H-1/2}\sigma\sqrt{v}$, $\varepsilon \to 0$ is the approximation parameter and $H \in [1/2, 1)$ is the Hurst parameter.

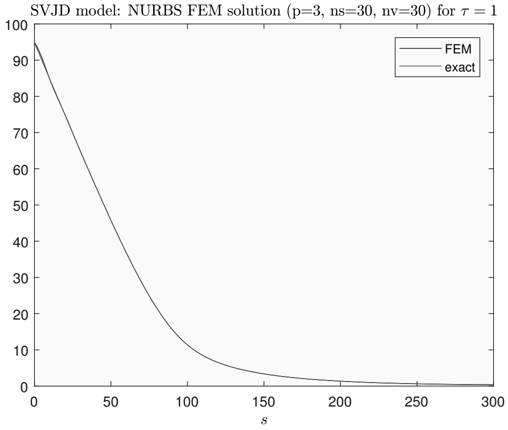
- If we take $H = 1/2$, we get the Bates (1996) model as a special case.
- If we take $H > 1/2$, the stochastic volatility process has the so called long memory.

Similarly we can proceed with the variation formulation and discretization.

FEM with NURBS basis functions

AFSVJD model - solution





Conclusion and further issues

Fitting results:

- multiplicity of knots can be easily used for example to describe also non-smooth payoff functions,
- rationality of NURBS give us much greater flexibility (compared to the standard B-splines) in describing complicated solutions of pricing equations,
- we can fit NURBS directly to the semi-closed pricing formulas.

Isogeometric analysis (FEM with NURBS basis functions) results:

- numerical solution of the Merton and AFSVJD model for European call/put option compared to the solution obtained by a semi-closed formula,
- very small number of space discretization steps can be used to obtain sufficiently accurate results,
- in general we get more precise results using higher order basis functions and using less discretization points.

Conclusion and further issues

Further and open issues:

- convergence for general PIDEs,
- more complex time-iterative schemes such as extrapolation schemes,
- pricing American options that leads to a solution of the partial integro-differential variational inequalities,
- rough models of fractional stochastic volatility (GACR Grant GA18-16680S, 2018–2020).

Thank you for your attention!

J. Pospíšil and V. Švígler (2017). *Isogeometric analysis in option pricing.* Manuscript resubmitted 04/2018.