Measuring Market Risk with Entropy

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Simply speaking, entropy is measure of disorder, uncertainty or surprise.

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What is entropy? (2)

Branch	Measure of…	Low entropy	High entropy
Thermodynamics	…particle disorder		
Information Theory	…message uncertainty or surprise	*“Sun will rise tomorrow”*	*“It will be sunny or rainy tomorrow”*
Risk Management	….market volatility or surprising P&L outcome		

Images source: pixabay.com

Motivation: Why entropy?

1. No assumptions about underlying distribution
2. Portfolio diversification leads to decrease in entropy by definition – subadditivity (*not true for VaR*)
3. More robust than standard deviation
4. Capped for distributions on finite interval (*not true for standard deviation*)
5. Always exists *(not true for standard deviation, e.g. Cauchy or other fat-tail distributions)*
6. Easy to interpret as measure of surprise

Mathematical model of entropy (1)

- Consider two possible outcomes of experiment with probabilities *p* and *1-p*

- If p = 0 or p = 1 there is no uncertainty:

  only one outcome is possible and it always occurs

- However for p = ½ uncertainty is at maximum:

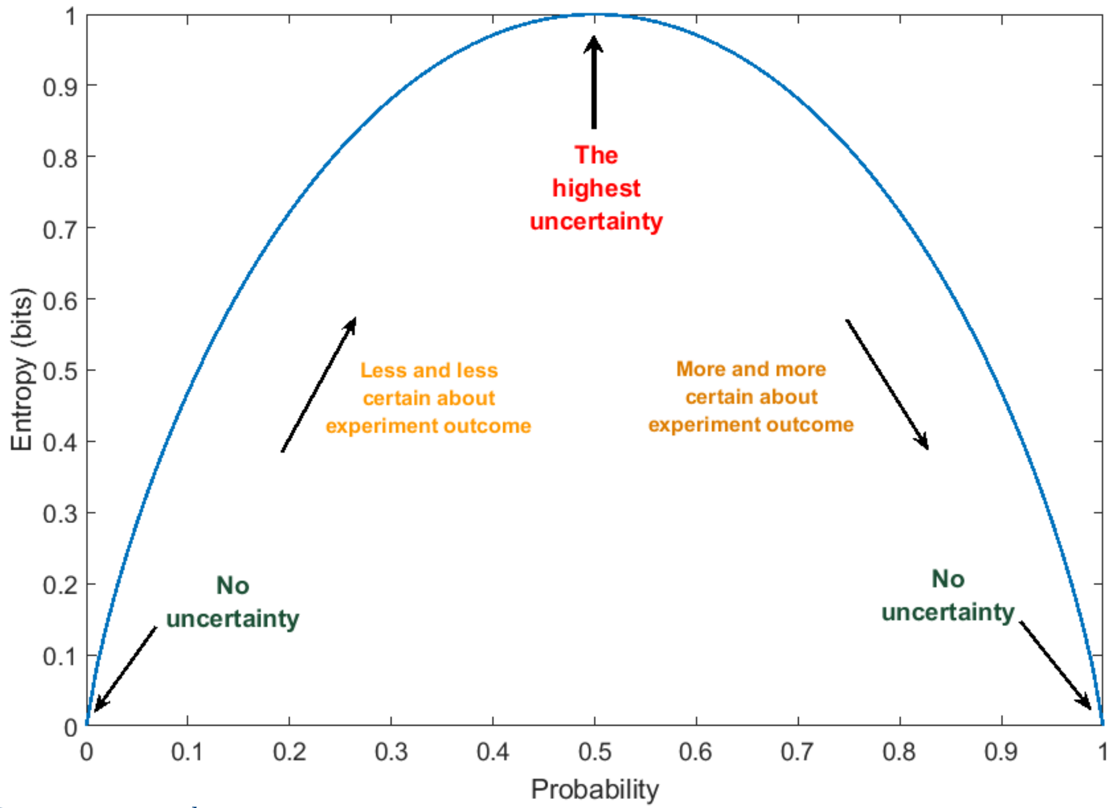
  50:50 $\Rightarrow$ no idea which outcome more likely to occur

- We are looking for function f(*p*) fulfilling
  - f(0) = f(1) = 0
  - Maximum occurs for p = 0.5

Mathematical model of entropy (2)

- One possible function is Shannon entropy:

$$H = -[p\log_a(p) + (1-p)\log_a(1-p)]$$



Source: own work

Notes:

1) *a = 2 for graph ⇒ entropy is expressed in bits (units will be discussed later)*
2) *0 log 0 is defined as 0 (i.e. value of p log(p) limit for p approaching zero)*

Mathematical model of entropy (3)

- Shannon entropy can be generalized for:

1. Discrete distributions (incl. distributions with $n \to \infty$)

$$H = -\sum_{i=1}^{n} p_i \log_a(p_i)$$

2. Continuous distributions ("reduced entropy")

$$H = -\int_R f(x) \log_a [f(x)] dx$$

Note: Reduced entropy can be negative. Lowest value (i.e. "no uncertainty") is $-\infty$.

Entropy and perfect certainty (1)

- Perfect certainty $\Rightarrow$ obviously zero entropy

- Inspired by 3rd law of thermodynamics:

  "Entropy of every system at absolute zero can be taken to be equal to zero"

- Shannon entropy of discrete distribution is zero by definition, but this is not the case for reduced entropy

- Total entropy of continuous distribution is in fact given by

$$H = -\int_R f(x) \log_a [f(x)] dx - \log_a [\Delta x]$$

Entropy and perfect certainty (2)

- Term $-\log_a[\Delta x]$ is "residuum" of switching from discrete to continuous realm
- Term approaches infinity as "delta" becomes "dee"
- To have H = 0, integral has to approach minus infinity for perfect certainty
  (!intuitive explanation, not mathematically fully correct!)
- **Reduced entropy** should be **used only for**:
  - Peer comparison
  - Time comparison
- **Overall, sufficient for use in finance**

Entropy units

- Entropy unit name depends on logarithm base used
  - $a = 2 \Rightarrow$ **bit**
  - $a = 3 \Rightarrow$ **trit**
  - $a = 10 \Rightarrow$ **dit**
  - $a = e\ (\approx 2.71) \Rightarrow$ **nat**
- Bits and nats are often used because…
  - …bits are usual measure of information content (*entropy of x bit means that one has to use x binary numbers for message encoding*)
  - …nats are connected with natural logarithm
- Entropy can be converted from base *a* to base *b* by dividing by $\log_a b$

How to calculate entropy?

1. Histogram estimator:
   - Based on definition for discrete case

$$\hat{H} = -\sum_{i=1}^{k} \frac{n_i}{n} \log_a\left(\frac{n_i}{n}\right) + \log_a(h)$$

Correction to bin width

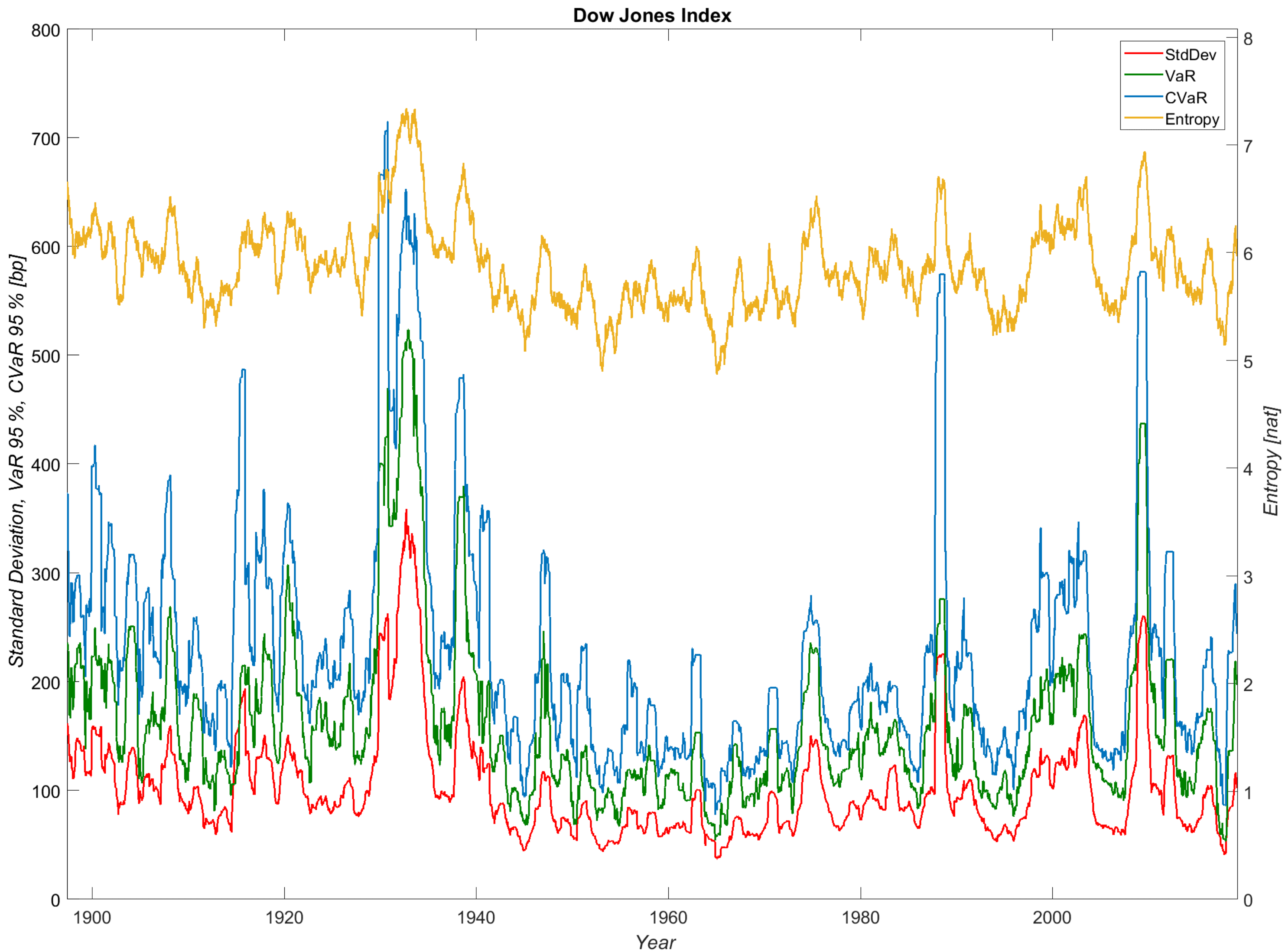
   - h…width of histogram bins, k…number of bins
     n_i…# observations in i^{th} bin, n…total # of observations
2. Kozachenko-Leonenko estimator (1D data):

$$\hat{H} = \frac{1}{n}\sum_{i=1}^{n} \log_a(r_i) + \log_a[2(n-1)] + \gamma$$

   - r_i is distance of observation to its nearest neighbour:
     $r_i = \min\{a_{i+1} - a_i; a_i - a_{i-1}\}$ for sorted observations a_i
   - If $r_i = 0$ then $r_i := 1/\sqrt{n}$
   - $\gamma \approx 0.5772156649$ (Euler-Mascheroni constant)

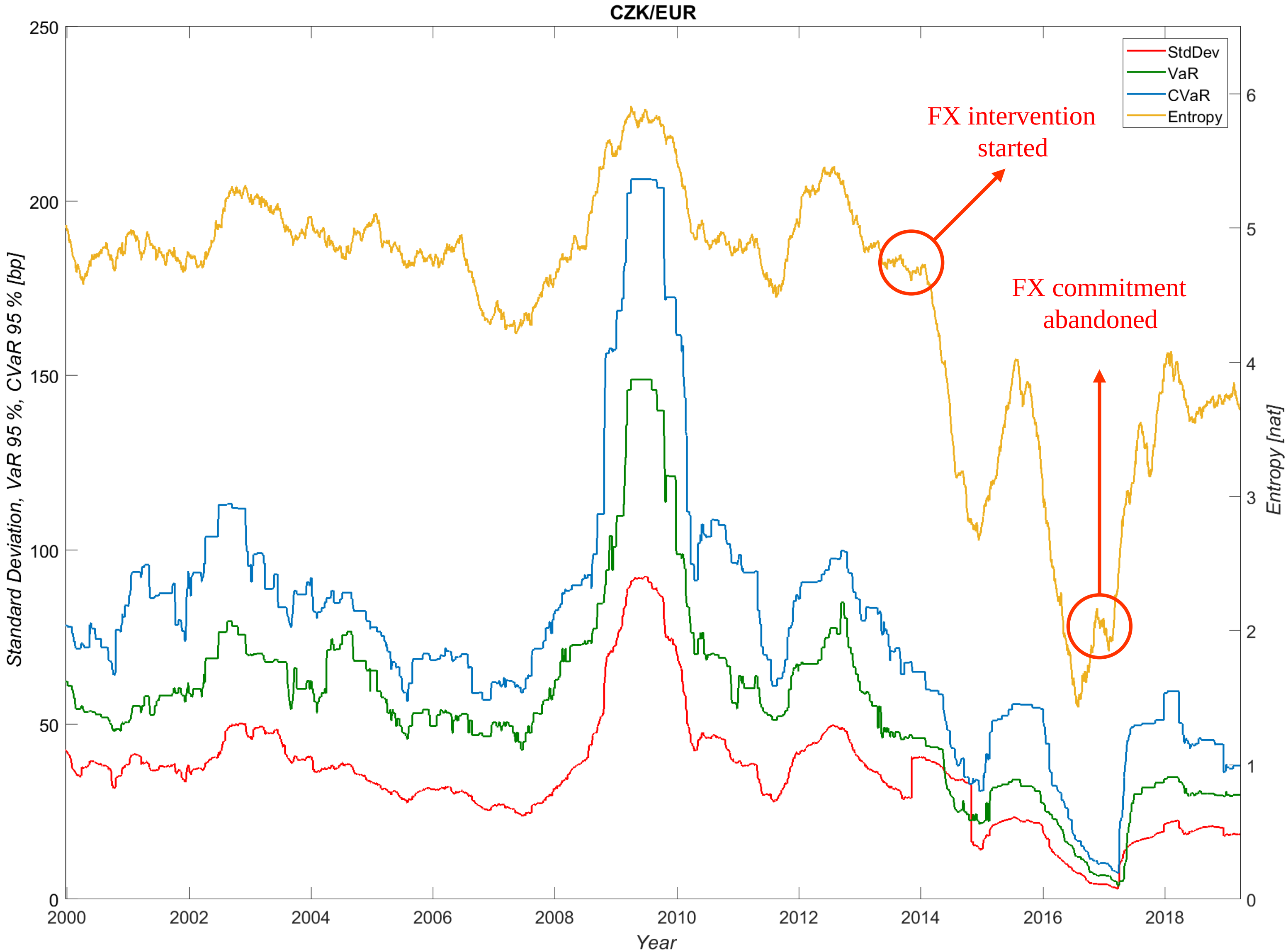
- Note: observation is actual value of e.g. P/L

Results – US equity markets



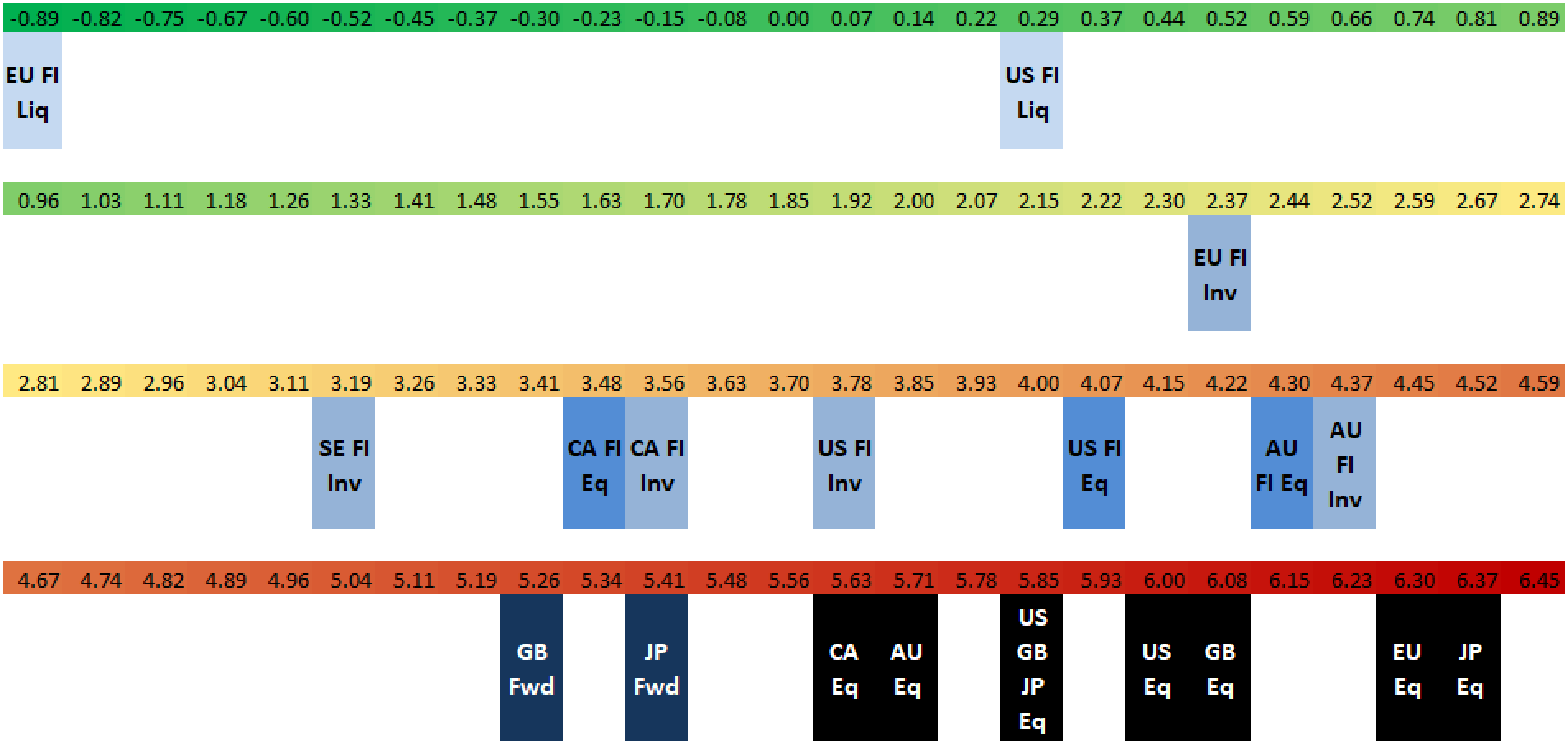
Source: Bloomberg, own calculation

Results – EUR/CZK



Source: Bloomberg, own calculation

Results – actual portfolios (03/2019)



Source: own calculation

legend:

FI…Fixed Income *Eq…Equity (shares)*

Liq…Liquidity ptf *Inv…Investment ptf*

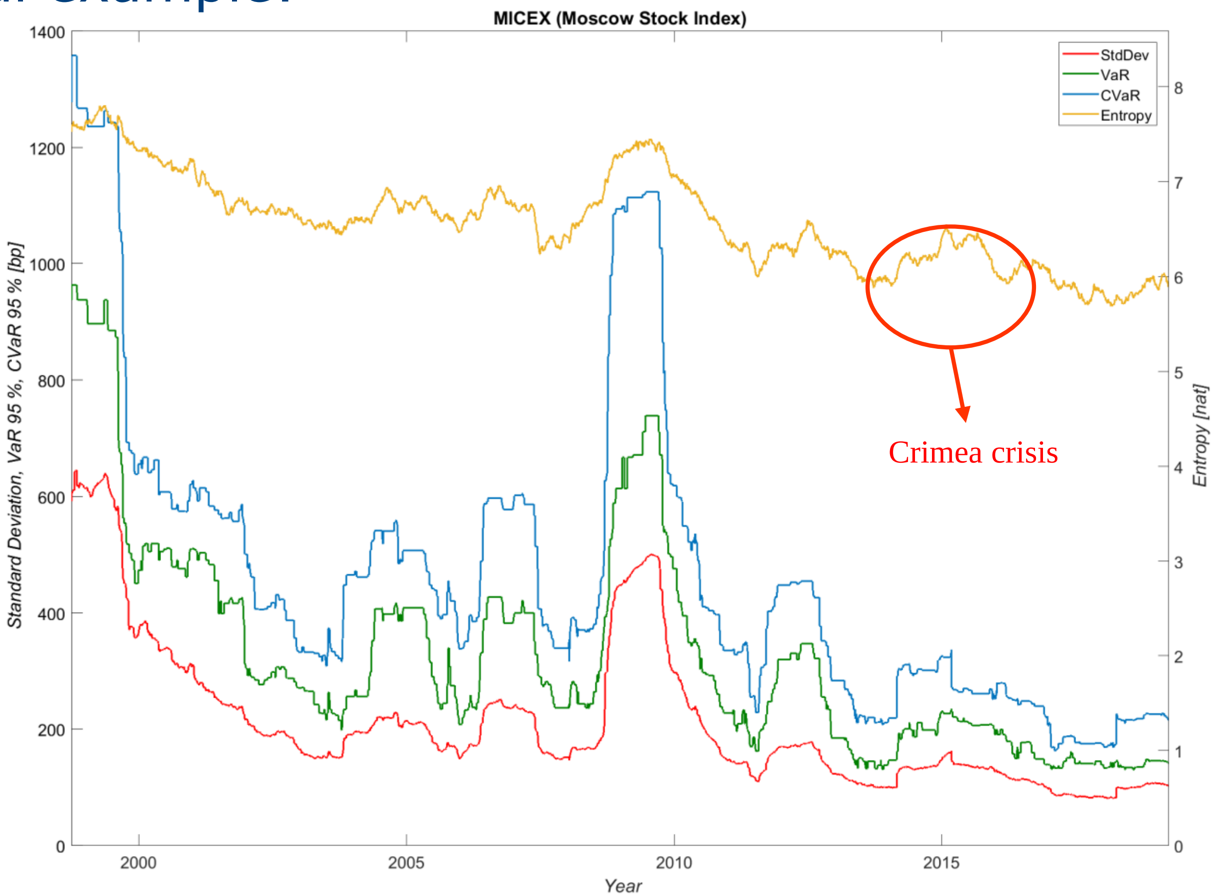
Fwd…Artificial cash position (currency overlay)

Results – entropy and behavioural finance (1)

- Entropy is more robust than standard deviation
  - less susceptible to outliers
  - $\Rightarrow$ less "overreaction" of market
- Theoretical example:
  - 10,000 random numbers distributed according to N(0,1) $\Rightarrow$ std. dev. is 0.979, entropy 1.387
  - Add 8 outliers: -7, -4.5, -4.5, -3, 3.9, 5, 5, 5.1
  - $\Rightarrow$ std. dev. is 0.987 (increase 0.8%), entropy 1.393 (increase 0.47%)
- Since entropy does not change as rapidly as standard deviation, investors should not overreact
- As a result, markets could be calmer, with only "shallow" crises

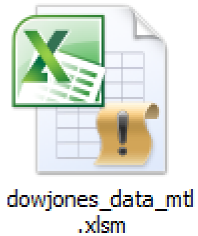
Results – entropy and behavioural finance (2)

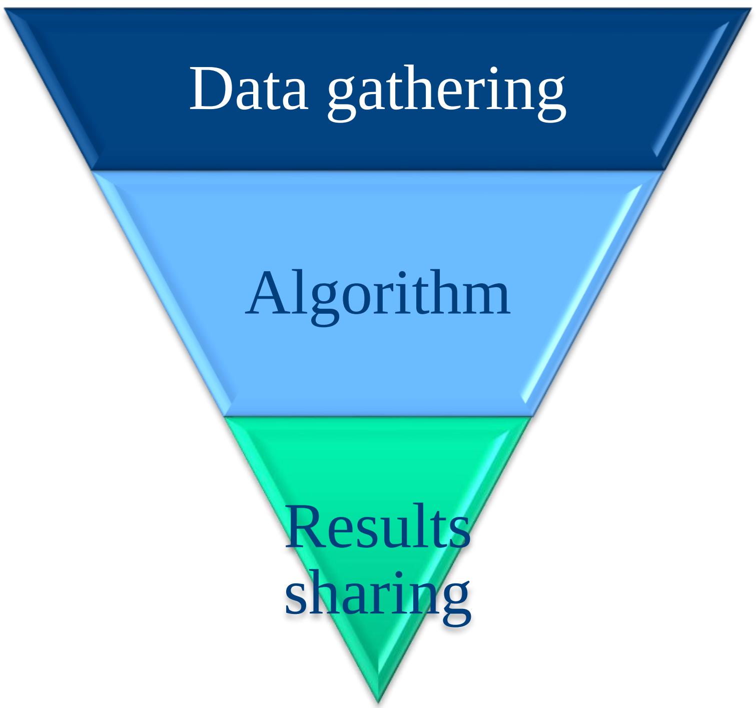
- Real example:



Source: Bloomberg, own calculation

- **Hypothesis**: Had investors followed entropy instead of std. dev, VaR or CVaR, sell-off of Russian equities would not have been so rapid after Crimea crisis outbreak





From raw Excel data…

```
T = readtable
    (inDataFileName,'sheet',sheetName);
TT = table2array(T(:,1));

TT(k,3) = std(X); %Standard Deviation
TT(k,4) = prctile(X,100 - alpha); %VaR
TT(k,5) = mean(X(X <= TT(k,4))); %CVaR
TT(k,6) = EntropyEstimationKL(X); %Entropy estim.

writetable(timetable2table(T), outDataFileName,
   'FileType','spreadsheet','Sheet','RiskMetrics')

h = plot(T.Time, table2array(T(:,3:end-1)));
savefig([outFigName, '.fig']); %MatLab figure
print([outFigName, '.png'],'-dpng'); %PNG file
```

Appendix C

…to graphic output for sharing

Conclusion

- Entropy is measure of disorder, surprise and information content
- Can therefore be used as measure of market risk
- Main advantages of entropy are:
  - Independence of underlying distribution
  - Always exists (stdDev does not)
  - Able to measure diversification correctly (VaR does not)
  - Easily to interpret as “level of surprising results”
  - More robust than e.g. standard deviation
- **Hypothesis:** using entropy can lead to calmer markets

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Appendix A: Shannon entropy is subadditive risk measure (1)

- Consider two portfolios A and B with *m* and *n* possible P/L outcomes:

$$A = \begin{pmatrix} A_1 & \dots & A_m \\ p_1{}^A & \dots & p_m{}^A \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & \dots & B_n \\ p_1{}^B & \dots & p_n{}^B \end{pmatrix}$$

- Combine these two portfolios into one A∗B, there are *mn* possible P/L outcomes (intersect):

$$A * B = \begin{pmatrix} A_1 + B_1 & \dots & A_1 + B_n & A_2 + B_1 & \dots & A_2 + B_n & \dots & \dots & A_m + B_n \\ \pi_{11} & \dots & \pi_{1n} & \pi_{21} & \dots & \pi_{2n} & \dots & \dots & \pi_{mn} \end{pmatrix}$$

- Entropy of combination is

$$H(A * B) = -\sum_{i=1}^{m}\sum_{j=1}^{n} \pi_{ij} \log_a \pi_{ij} = -\sum_{i=1}^{m}\sum_{j=1}^{n} p_j{}^B p_i{}^{A|B=j} \log_a \left[p_j{}^B p_i{}^{A|B=j} \right]$$

where $p_i{}^{A|B=j}$ is conditional probability of i-th outcome in portfolio A if j-th outcome in B occurred simultaneously

Appendix A: Shannon entropy is subadditive risk measure (2)

- After some algebra it is possible to write

$$H(A*B) = H(A|B) + H(B)$$

where

$$H(A|B) = -\sum_{j=1}^{n} p_j^{\;B} \sum_{i=1}^{m} p_i^{\;A|B=j} \log_a \left[p_i^{\;A|B=j} \right]$$

Conditional entropy of A
(entropy of A if B influences A)

Partial entropy of A
(entropy of A if j-th outcome in B occurred)

- Naturally, if A and B are independent:
  1. $H(A|B) = H(A)$
  2. Any knowledge about B cannot be used for decreasing uncertainty in A, so H(A) should intuitively be maximum of H(A|B)

Note: this statement can be proven rigorously with Jensen inequality

Appendix A: Shannon entropy is subadditive risk measure (3)

- Combining…
  1. $H(A * B) = H(A|B) + H(B)$
  2. $H(A|B) \leq H(A)$
- …we have $\boldsymbol{H(A * B) \leq H(A) + H(B)}$
- **This means that combination of two portfolios together reduces risk**
  ⇒ entropy is subadditive risk measure
- This can be proven for continuous distributions as well

Appendix B: Other types of entropy

- Tsallis entropy:

$$H_T = \frac{k}{q-1}\left(1 - \sum_{i=1}^{n} {p_i}^q\right)$$

- Renyi entropy:

$$H_R = \frac{k}{1-q}\log_a\left(\sum_{i=1}^{n} {p_i}^q\right)$$

- For *q* approaching 1, both of them become Shannon entropy

- For q = 0 Renyi entropy becomes entropy in thermodynamics sense (Clausius/Boltzman definition):

$H = k\log_a n$ or $S = k\log_a \Omega$ in thermodyn. notation

Appendix C: MatLab code for Kozachenko-Leonenko estimator

```
function E = EntropyEstimationKL(input)
  n = length(input);

  %distance to the nearest neighbour
  input = sort(input);
  r = zeros(1,n);
  r(1) = input(2)-input(1);
  r(2:end-1) = min(input(3:n) - input(2:n-1), input(2:n-1) - input(1:n-2));
  r(n) = input(n)-input(n-1);

  %elimination of denegerated values
  r(r==0) = 1/sqrt(n);

  %actual estimation of entropy
  E = (1/n)*sum(log(r))+log(2*(n-1))+0.5772156649;
```

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