SOLVING THE TRANSIENTS EVENT IN ELECTRIC CIRCUITS USING A MATHEMATICAL MODEL OF DIFFERENTIAL EQUATIONS

M. Liška
Slovak University of Technology in Bratislava, Faculty of Electrical Engineering and Information Technology, Ilkovičova 3, 812 19 Bratislava 1
martin.liska@stuba.sk,

Abstract

Electrical circuits are systems that can be described in different ways using differential equations of first, second and upper order. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy. If the circuit contains storage elements such as capacitors and inductance, these circuits can be described as integral-differential equations. Analytical solution of such circuits in complex networks is very difficult and lengthy. To calculate the circuit state variables (voltages and currents) can be used those programs in which the electrical circuit is created with a simple drag & drop system "and connecting graphical blocks representing directly individual electrical devices. This works some a desktop application such as the EMTP-ATP or Matlab/Simulink. Advantage of solution lies in the fact, that the algorithm changes are implemented through this application, changing, deleting and adding blocks of the circuit. The disadvantage of this approach is that it avoids the need to create a mathematical model of the circuit. Physical basis for a deeper understanding of the behavior of the circuit in steady state, but also for the transient events is necessary. This paper deals with the calculation of state variables of the electrical circuit using a mathematical model consisting of differential equations.

1. Theoretical part:

Circuit differential equations are essentially based on two Kirchhoff laws and their creating may use general methods, but most often used method of loop currents or node-voltage method, or modified nodal voltage method. Creating equations of concrete branches of the circuit is then based on the fundamental relationship between the current state variables at the individual circuit elements:

Resistance: $$u(t) = R * i(t) \tag{1}$$

Capacity: $$u(t) = \frac{1}{C} * \int i(t)dt = u_{0+} + \frac{1}{C} * \int_0^t i(t)dt, \quad i(t) = C * \frac{du(t)}{dt} \tag{2}$$

Inductance: $$i(t) = \frac{1}{L} * \int u(t)dt = i_{0+} + \frac{1}{L} * \int_0^t u(t)dt, \quad u(t) = L * \frac{di(t)}{dt} \tag{3}$$

Using these relations based on a description above of the complex circuit the state variables are becoming describe as a set of integral-differential equations. Integral equations can be easily converted by derivate to the differential equations. The complex circuit is then described as a system of n linear or nonlinear differential equations with constant coefficients, respectively single differential equation of n-th order [1]:

$$\mathrm{a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots .. a_1 \frac{dx}{dt} + a_0 x = y(t)} \tag{4}$$

The solution of the diferential equation consist from the homogenous equation X0(t) and particular solutions Xp(t).

$$X(t) = X_o(t) + X_p(t) \tag{5}$$

Homogenous equation:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots .. a_1 \frac{dx}{dt} + a_0 x = 0 \tag{6}$$

Its general solution depends only on the characteristics of the circuit without any independent sources. However, it is fundamentally affected by the initial energy state of the circuit, ie sizes of the accumulated energy in capacitors and coils at the beginning of a solution, at t=0s. The character of the solution equation is the given by the roots λ_1, λ_2,....λ_n of so characteristic equation, which is a polynomial equation form:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \cdots . + a_1 \lambda + a_0 = 0 \tag{7}$$

If the polynomial roots are simple and different of each other, the solution of homogeneous differential equations is given by linear combination of exponential functions of type exp (λ_{kt}), ie

$$X_o(t) = \sum_{k=1}^{n} K_k e^{kt} \tag{8}$$

Where K_1, K_2.....K_n are integration constants whose values determine the specific initial conditions in the system.

When operating in the circuit periodic or direct voltage and current sources, the circuit reaches after the transient the stationary or periodic steady state expressed by homogeneous equation X0(t). Steady state is thus expressed precisely by particulate solution.

2. Solving the circuit state variables using differential equation – mathematical model of simply electrical circuit given by linear differential equation 2-th order:

The figure (Fig. No. 1) shows the scheme of simple RLC circuit supplying with DC voltage source voltage Us and the equivalent circuit model created in software Matlab / Simulink. At the time t = 0, the switch closes and the circuit is connected to the voltage source. In the circuit we are interested in state variables, the voltage on capacitor uc(t) and the circuit current i(t). Other variables such as voltage for inductance and resistance can be easily determined from known relationships. Sizes of circuit elements are: Us = 1V, R = 15Ω, L = 20mH, C = 333μF.

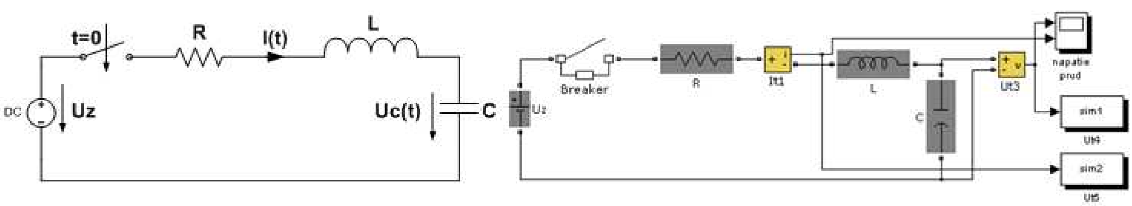

Fig. 1 – Electrical circuit scheme and its equivalent model in Matlab/Simulink

After writing II. Kirchhoff's law and an expression of current via capacity voltage we will obtain a homogeneous linear differential equations 2-th. order with constant coefficients, which reflects the voltage loop circuit:

$$\frac{d^2 u_c(t)}{dt^2} = \frac{U_s}{LC} - \frac{R}{L} \frac{du_c(t)}{dt} - \frac{1}{LC} u_c(t) \tag{9}$$

Solving of this differential equation we get the capacitor voltage uc(t). The current circuit is calculated via derivation and multiplying the capacitor voltage and the value of capacitor C. This differential equation is thus a mathematical model of that circuit. The solution of this mathematical model is performed in the software Matlab / Simulink in two ways (Fig. 2). Via gradual integration and using the transfer function G(s) entered in block Transfer Fcn.

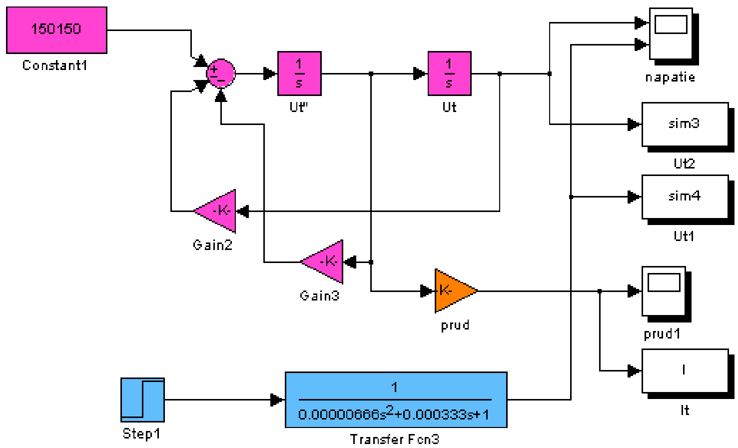

Fig. 2 – scheme of mathematical model for solving the differential equation (gradual integration and transfer function)

Transfer G (s) of general dynamic system (in this case the circuit) is defined as the proportion of the Laplacian image of output signal and Laplacian image of input signal at zero initial conditions:

$$G(s)\frac{Y(s)}{U(s)} \tag{10}$$

Where s is an argument of Laplace transformation. Y(s) is an Laplacian image of output signal and U(s) is the Laplacian image of input signal. According to these, the transfer function G(s) of the electrical circuit shown at the figure 1 is:

$$G(s) = \frac{U_s}{(LC)s^2+RCs+1} \tag{11}$$

Simulation Results:

The following figures shows the capacity voltage and current in the circuit .Figure 3 shows these state variables obtained from the circuit at the Figure 1 and Figure 4 shows the values obtained from the mathematical model, which solution is obtained from the scheme at the Figure 2.

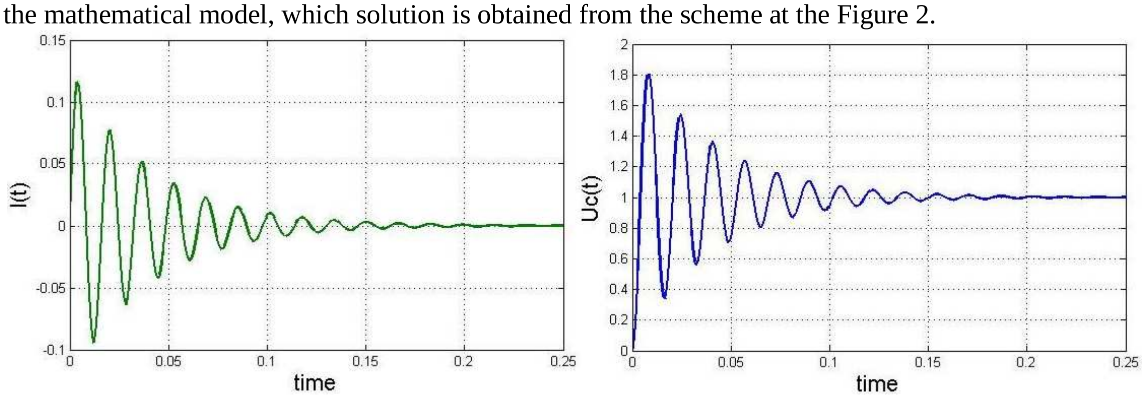

Fig. 3 – The circuit current and the capacity voltage according to scheme at the fig. 1

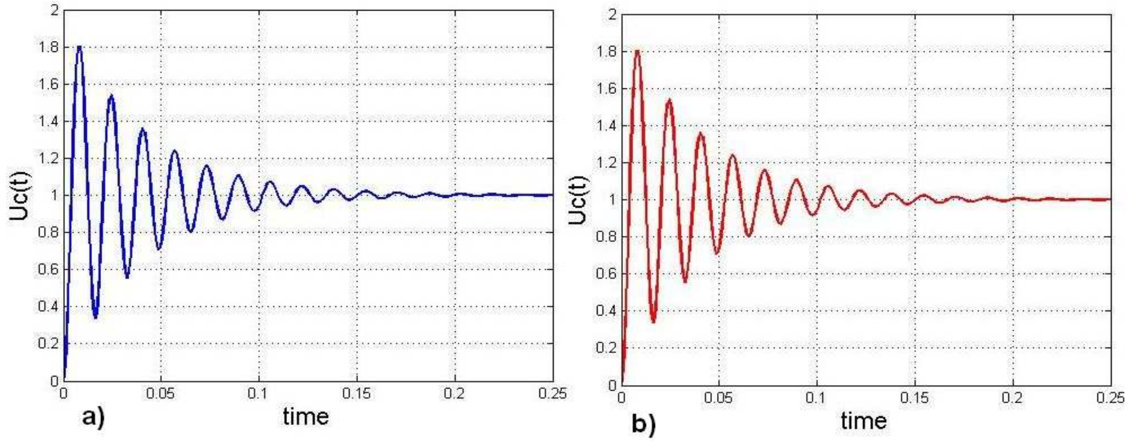

Fig. 4 – The capacity voltage according the scheme at the fig. 2: a) model of gradual integration b) model of transfer function

If we consider that the voltage (V) and current (A) according to Fig. 3 (respectively Fig.1) as a reference, the comparison of time course and value of voltage and current obtained from the model in Fig. 2, confirms that the mathematical model consists of differential equation is created correctly. It means, both ways of creating mathematical model of that circuit (via gradual integration and transfer function) are correct.

3. Solving more complex circuits using the state equation and output equation system:

In the first part, the mathematical model was created using an one 2-th order differential equation order to describe the serial RLC circuit. For more complex circuits it is possible to create the mathematical model form in the same way. This method, however, takes longer and for more more complex circuits is not practice. An appropriate method is to create a model using the system state equation and the output equation of such complex system. The system state equation can by given by this formula [2]:

$$x = A(t)x + B(t)u \tag{12}$$

and the output equation

$$y = C(t)x + D(t)u \tag{13}$$

T - invariant systems (systems whose parameters do not change with time) can be described by equations, where A, B, C, D are matrices of constants [2]:

$$x = Ax + Bu$$

$$y = Cx + Du \tag{14}$$

The variable x represents the state of the system , u is an system input and y is the output. The dynamic equation can be expressed as a simple structure (Fig. 6)

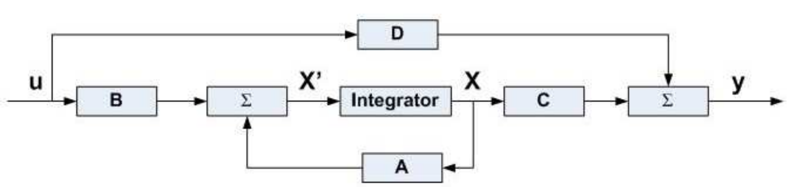

Fig. 6 - structured scheme of dynamic system

A – system dynamics matrix
B – matrix of inputs
C – matrix of outputs
D - direct exposure matrix input to output

The electrical circuit scheme:

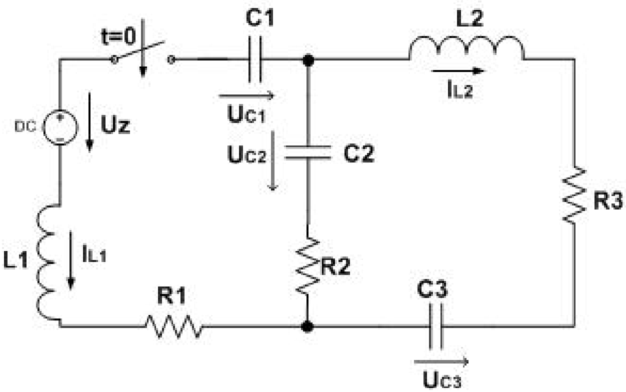

Fig. 7 – electrical circuit scheme

At the time t = 0 the switch will close and the circuit transient action will occur. The task is to determine the time courses of state variables: iL1(t), iL2(t), uC1(t), uC2(t), uC3(t). The values of individual circuit components are Uz = 15V, R1 = 120Ω, R2 = 140Ω, R3 = 410Ω, C1 = 45nF, C2 = 210μF, C3 = 1250μF, L1 = 15mH, L2 = 22mH,

The circuit as shown above (Fig. 7) it is possible to describe with systems of differential equations:

$$\begin{aligned}
\frac{di_{L1}}{dt} &= -\frac{R_1+R_2}{L_1} i_{L1} + \frac{R_2}{L_1} i_{L2} - \frac{1}{L_1} u_{C1} - \frac{1}{L_1} u_{C2} + \frac{U_z}{L_1} \\
\frac{di_{L2}}{dt} &= \frac{2R_1+R_2}{L_2} i_{L1} - \frac{R_2+R_3}{L_2} i_{L2} + \frac{1}{L_2} u_{C2} - \frac{1}{L_2} u_{C3} \\
\frac{du_{C1}}{dt} &= \frac{1}{C_1} i_{L1} \\
\frac{du_{C2}}{dt} &= \frac{1}{C_2} i_{L1} - \frac{1}{C_2} i_{L2} \\
\frac{du_{C3}}{dt} &= \frac{1}{C_3} i_{L12}
\end{aligned} \tag{15}$$

Individual dynamic matrix equation system then have the form:

$$A = \begin{pmatrix} -\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & -\frac{1}{L_1} & -\frac{1}{L_1} & 0 \\ \frac{2R_1+R_2}{L_2} & -\frac{R_2+R_3}{L_2} & 0 & \frac{1}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C_1} & 0 & 0 & 0 & 0 \\ \frac{1}{C_2} & -\frac{1}{C_2} & 0 & 0 & 0 \\ 0 & \frac{1}{C_3} & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \frac{1}{L_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{16}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Dynamic equations can be in solved in the program Matlab / Simulink using the block State-Space and the values of matrices A, B, C, D to enter directly into the block. Image (Fig.8) shows the block diagram of the model:

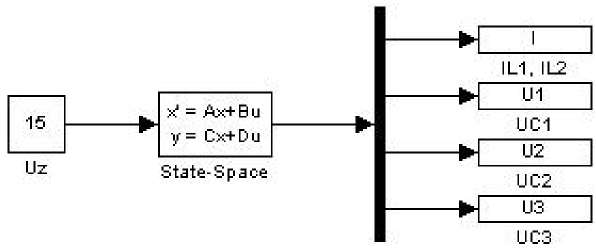

Fig. 8 - Block diagram of dynamic system specified by the block "State-Space"

The following figures are shown the courses of solving state variables:

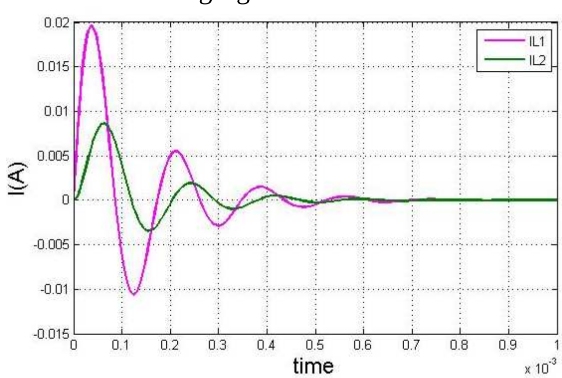

Fig. 9 – course of currents IL1 a IL2

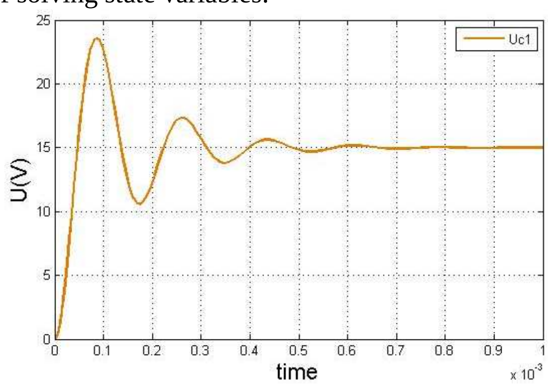

Fig. 10 – capacity voltage on C1

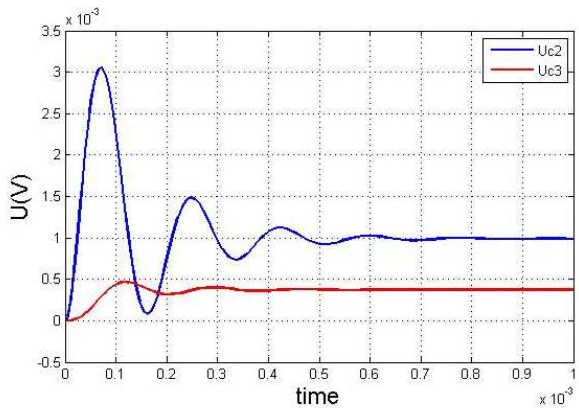

Fig. 11 – capacities voltage on C2 and C3

At the time $t \to \infty$ (steady state) no current flows thru circuit. Therefore, voltage declines on the resistance R2 and R3 are zero. It follows that the voltage on the capacitors C2 and C3 should be the same in steady state. In Fig. 11, these voltages are not equal, because of the simulation time 1,10-3s is not sufficient to balance the energy of both capacitors. Figure 10 confirms the correctness of the voltage capacitors C2 and C3 calculation, where the simulation time is prolonged to 0,4 s

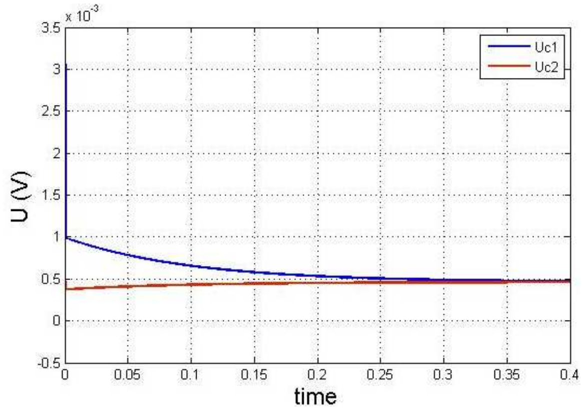

Fig. 11 – capacities voltage on C2 and C3, prolonged simulation time to 0,4s

The voltage on both capacitors at the time t = 0,4s stabilized at the value nearly to 0,5 mV. After editing circuit in Figure 7. At the time $t \rightarrow \infty$ and writing II. Kirchhoff's law is clear, that if the voltage on capacitors C2 and C3 is 0,5 mV, the voltage on C1 capacitor should have a value of 15V-0, 5mV. On this basis, the figure 10 with capacity voltage course of C1 confirms the correctness of the solution.

4. Conclusion:

The theory of differential equations, which is normally available between the scientific literature, results conditions of stability systems, the values of damping, time constants etc. Solving the transient phenomena in power or low voltage electrical circuits via using a mathematical model of differential equations, allowing a better understanding of the behavior of the model. For example, the behavior of the system (stability, damping, etc) is possible to change via changing constants by individual derivations – changing the values of electrical components. To understand this and translated into practical usage in the process of designing circuits, it can be easily achieved when individual circuits components are not modeled by RLC blocks , but using their relationship between voltage and current.

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0546-07. Author is grateful for support to the Grant VEGA No. 1/0687/09.

References

[1] SEDLÁČEK, J., VALSA, J., Elektrotechnika II, Fakulta elektrotechniky a komunikačných techológií, Vysoké učení technické v Brně

[2] KARBAN, P., Výpočty a simulace v programech Matlab a Simulink, . vyd. Brno : Computer Press, a.s. 2006. ISBN 80-251-1301-9