MATLAB-IMPLEMENTED NUMERICAL ALGORITHMS FOR THE COMPUTATION OF STRUCTURAL ELEMENTS IN LINEAR MULTIVARIABLE MODELS OF PHYSICAL SYSTEMS

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Abstract

Physical systems represented in terms of linear time-invariant multivariable models are endowed with certain structure which is important for the analysis and synthesis of control strategies. Structural elements of the two widely used representations —autoregressive representation with external variables on equal footing (in the frequency domain) and state-space representation with static constraints (in the time domain)— are computed using the tools of numerical linear algebra of constant matrices.

Keywords: Linear systems, Subspace methods, State feedback

This is a preliminary version of a paper to appear elsewhere.

1 Introduction

The problems described in this paper are of specific interest in linear system theory but are analyzed using tools that are more numerical linear algebra oriented than the mainstream of systems and control literature. A basic problem is the study of the equation

$$P(\lambda)w(t) \quad = \quad 0 \tag{1}$$

where $P(\lambda)$ is a singular[1] polynomial matrix with constant coefficients. The operator λ may either be the differential operator $\frac{d}{dt}$ or the shift operator z and $w(t)$ is then a vector-valued time function or time series, respectively.

Structural elements of polynomial matrices – which are a special case of rational matrices – were described in the algebraic theory, e.g. by McMillan (1952) and Forney, Jr. (1975), based on partial results in earlier literature, e.g. Smith (1873), Kronecker (1890), Gantmacher (1953). In the most general case of a singular polynomial matrix $P(\lambda)$, the structural elements consist of zeros, poles (at infinity), and the left and right null spaces. These structural elements are known to play a fundamental role in practical problems of coding theory (Forney, Jr., 1975), network theory (McMillan, 1952), control theory (Wonham, 1979) and other related fields (Kailath, 1980). A nonsingular[2] polynomial matrix $P(\lambda)$ has no left and right null space but the zeros appear in a geometric structure which plays a fundamental role in some applications, e.g. Kraffer and Kwakernaak (1997).

System-theoretic interpretations and numerical problems subject to this paper are related to general rational matrices with the field of real numbers $\mathbb{R}$ as coefficient space, implying the occurence of complex numbers $\mathbb{C}$ in our definitions of poles and zeros. Consequently, we prefer

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[1]A polynomial matrix is singular if it is not square and invertible.

[2]A polynomial matrix $P(\lambda)$ is nonsingular if $\det P(\lambda) \not\equiv 0$, i.e. $P(\lambda)$ will be nonsingular for almost all values of λ, except those that make $\det P(\lambda) = 0$.

to embed $\mathbb{R}$ in $\mathbb{C}$. By $\mathbb{C}[\lambda]$ we denote the ring of polynomials in λ over $\mathbb{C}$ and by $\mathbb{C}((\lambda^{-1}))$ the field of rational functions in λ over $\mathbb{C}$. The ring of rational functions bounded at $\lambda = \infty$ is called *proper* and denoted $\mathbb{C}[[\lambda^{-1}]]$, its subset with vanishing constant term is called *strictly proper* and denoted $\lambda^{-1}\mathbb{C}[[\lambda^{-1}]]$. An invertible matrix in $\mathbb{C}^{n\times n}[\lambda]$ is called *unimodular* whereas an invertible matrix of $\mathbb{C}^{n\times n}[[\lambda^{-1}]]$ is called *biproper*. It is easy to see that matrices in $\mathbb{C}^{n\times n}[\lambda]$ are unimodular if and only if their determinant is a constant, different from zero (Gantmacher, 1953). Two matrices $R_1, R_2 \in \mathbb{C}^{m\times n}((\lambda^{-1}))$ are said to be equivalent when there exist unimodular matrices $M \in \mathbb{C}^{m\times m}[\lambda]$ and $N \in \mathbb{C}^{n\times n}[\lambda]$, respectively, such that $MR_1N = R_2$. Indeed, this is an equivalence relation since the inverse and the product of unimodular matrices are again unimodular.

In numerical analysis this led, among other things, to the problem of computing the eigenstructure of a polynomial matrix (Van Dooren and Dewilde, 1983). In system theory one is rather interested in algebraic properties like stability, state-space descriptions, etc. (Rosenbrock, 1970) and their physical applications e.g. optimal and robust control, filtering, etc. (Kwakernaak and Sivan, 1972), (Wonham, 1979), (Stoorvogel, 1992). Meanwhile more differential-equation-oriented approaches were studied in linear system theory by Blomberg and Ylinen (1983) and Willems (1995). In that area of interest a system is looked at as a set of trajectories yielding an algebraic description of their generator (Blomberg and Ylinen, 1983). A more general case where the inputs and outputs are considered on equal footing has been pioneered by Willems; see Willems (1995) for an informal sketch and references. For a general system of differential equations viewed as a generator of a set of trajectories for external variables on equal footing, a complete list of (externally) equivalent operations is given in Schumacher (1988).

2 Column reduced polynomial matrices, state feedbacks, and maximally unobservable subspaces

The notion of column reducedness is widely used in system theory. The basic result is the following (Wonham, 1979):

Theorem 2.1 (column reduced polynomial matrix) *Let $P \in \mathbb{C}^{p\times m}[\lambda]$. Then there exists a unimodular matrix $U \in \mathbb{C}^{m\times m}[\lambda]$ such that $PU = \begin{bmatrix} P_\mathrm{r} & 0 \end{bmatrix}$ with $P_\mathrm{r} \in \mathbb{C}^{p\times r}[\lambda]$ a full column rank, column reduced matrix with column degrees $k_1, k_2, \ldots, k_r$ decreasingly ordered. The column degrees are uniquely determined, although U is not, and are called the column indices of P.*

Corollary 2.2 (Wiener-Hopf left factorizations) *Let $P \in \mathbb{C}^{p\times m}[\lambda]$ be the polynomial matrix in theorem 2.1. Then there exists a left factorization in the form $P = BDU$ with $U \in \mathbb{C}^{m\times m}[\lambda]$ unimodular, $B \in \mathbb{C}^{p\times p}[[\lambda^{-1}]]$ biproper, and*

$$D = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta = \mathrm{diag}(\lambda^{k_1}, \ldots, \lambda^{k_r}). \tag{2}$$

Proof: After a unimodular transformation of P into a column reduced P_r, we may write P_r as

$$\begin{aligned} P_\mathrm{r}(\lambda) &= P_\mathrm{rhc}\Delta(\lambda) + P_\mathrm{rlc}\psi(\lambda) \\ \psi^\mathrm{T}(\lambda) &= \text{block diag}\left(\begin{bmatrix} \lambda^{k_1-1} & \ldots & \lambda & 1 \end{bmatrix}, \ldots, \begin{bmatrix} \lambda^{k_r-1} & \ldots & \lambda & 1 \end{bmatrix}\right) \end{aligned} \tag{3}$$

or $\left(P_\mathrm{rhc} + P_\mathrm{rlc}\,\psi(\lambda)\Delta^{-1}(\lambda)\right)\Delta(\lambda)$ where the nonsingularity of P_rhc and the strict properness of $\psi(\lambda)\Delta^{-1}(\lambda)$ ensures that $P_\mathrm{rhc} + P_\mathrm{rlc}\,\psi(\lambda)\Delta^{-1}(\lambda)$ is bounded at infinity, together with its inverse.
□

If P is singular then the left factorization of P is obviously not unique. In the nonsingular case the factorization is also not unique but the freedom of nonuniqueness is smaller.

Remark 2.3 (non-unique factorizations, unique factorization indices) *A left factorization $P = B\Delta U$ of a nonsingular $P \in \mathbb{C}^{m\times m}[\lambda]$ is nonunique since $P = (BV)\Delta(\Delta^{-1}V^{-1}\Delta U)$*

describes another left factorization, generated by an element of a multiplicative group of unimoular matrices $V \in \mathbb{C}^{m\times m}[\lambda]$ *whose entries satisfy*

$$\begin{array}{rcll} u_{ij} & = & 0 & \text{if } k_i > k_j \\ \deg u_{ij} & \leq & k_j - k_i & \text{if } k_i \leq k_j \end{array}$$

A similar result may be obtained in the singular case to yield the uniqueness of the factorization indexes for any $P \in \mathbb{C}^{p\times m}[\lambda]$.

Remark 2.4 (row reducedness/degrees/indices, right factorizations) *Row reducedness, row degrees, row indices, Wiener-Hopf right factorizations, and the right factorization indices are defined analogously by considering the transpose of* P *and taking the transpose of the resulting quantities.*

In order to introduce conventional realization theory, let us consider a system of differential equations in the form

$$P_{\mathrm{r}}(\lambda)\xi(t) = u(t) \tag{4}$$

where $P_{\mathrm{r}} \in \mathbb{C}^{m\times m}[\lambda]$ is nonsingular and column reduced with nonzero column degrees

$$k_1 \geq k_2 \geq \cdots \geq k_m.$$

We may think of ξ as the partial state of a physical system whose output dynamics is described by

$$y(t) = Q_{\mathrm{r}}(\lambda)\xi(t), \tag{5}$$

$Q_{\mathrm{r}} \in \mathbb{C}^{p\times m}[\lambda]$ having its column degrees strictly less than the corresponding column degrees of P_{r} in order to have an eligible transfer function for the physical system, that is, $Q_{\mathrm{r}}P_{\mathrm{r}}^{-1} \in \lambda^{-1}\mathbb{C}^{p\times m}[[\lambda^{-1}]]$.

Having written P_{r} in the form (3), we may arrange (4) in the form

$$\Delta(\lambda)\xi(t) = -P_{\mathrm{r}hc}^{-1}P_{\mathrm{r}lc}\psi(\lambda)\xi(t) + P_{\mathrm{r}hc}^{-1}u(t) \tag{6}$$

in order to allow (4) a simple closed-loop interpretation: the closed loop consists of a dynamic gain $P_{\mathrm{r}hc}^{-1}P_{\mathrm{r}lc}\psi(\lambda)$, wrapped in a feedback around a system whose transfer function, input, and output are described by $\Delta^{-1}(\lambda)$, $\Delta(\lambda)\xi(t)$, and $\xi(t)$, respectively, the closed-loop input and output being described by $P_{\mathrm{r}hc}^{-1}u(t)$ and $\xi(t)$.

Apparently, Δ^{-1} describes the integrators required to set up a state-space realization of the loop. The integrators appear in m chains, each containing k_i integrators. In accord with Kelvin's method (Thomson, 1876), the state x of the realization may be chosen as

$$x(t) = \psi(\lambda)\xi(t), \qquad \dim x = \sum_{i=1}^{m} k_i$$

and the realization may be interpreted in terms of a state-space realization of

$$\xi(t) = \psi(\lambda)\Delta^{-1}(\lambda)v(t) \tag{7}$$

by wrapping the realization of (7) in a state feedback and applying a nonsingular transformation to the input coordinates of thus obtained loop. Indeed, in transfer function terms, application of

$$v(t) = P_{\mathrm{r}hc}^{-1}u(t) - P_{\mathrm{r}hc}^{-1}P_{\mathrm{r}lc}x(t) \tag{8}$$

to (7) recovers (4) while, in state space terms, application of (8) to a minimal state-space realization of (7) yields a minimal state-space realization of (4).

Perhaps the simplest minimal realization of (7) is described by (A, B, C) in the celebrated Brunovsky canonical form (Brunovsky, 1970):

$$\begin{aligned} A &= \text{block diag}\left\{ \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, k_i \times k_i, i = 1, \dots, m \right\} \\ B^{\mathrm{T}} &= \text{block diag}\left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, 1 \times k_i, i = 1, \dots, m \right\} \\ C &= I_n, \quad n = \textstyle\sum_{i=1}^m k_i \end{aligned} \tag{9}$$

as may be checked by direct calculation.

Remark 2.5 (state feedback) *Note that state feedback: does not alter* $Q_{\mathrm{r}}(\lambda)$*, the numerator polynomial matrix of a* $Q_{\mathrm{r}} P_{\mathrm{r}}^{-1} \in \lambda^{-1}\mathbb{C}^{p\times m}[[\lambda^{-1}]]$*; does not alter* P_{rhc}*, the highest column degree coefficient matrix of* $P_{\mathrm{r}}(\lambda)$*; can completely change* P_{rlc}*, the lower column degree coefficient matrix of* $P_{\mathrm{r}}(\lambda)$*.*

Remark 2.6 (input space transformation) *Note that the highest column degree coefficient matrix* P_{rhc} *may be scaled to an arbitrary (equally dimensioned) nonsingular matrix by a nonsingular transformation of the input space.*

Definition 2.7 (regular state feedback) *Given a state-space realization* (A, B, C)*, specified by a triple of constant matrices* A, B, C *such that*

$$\begin{aligned} \lambda x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

a regular state feedback around (A, B, C) *is defined by*

$$v(t) = Gu(t) + Kx(t)$$

with G *and* K *constant matrices,* G *nonsingular.*

Corollary 2.8 (feedback equivalence transformations) *The set of transformations defined on* (A, B, C) *by regular state feedback and similarity transformation defines an equivalence relation on* (A, B, C)*, the Brunovsky form (9) and the set of column degrees* k_1, k_2, $\dots$, k_m *being a canonical representative of an equivalence class and the set of invariants within the class, respectively.*

The fact that the above transformations do not affect controllability can be proved similarly to the scalar case: their application preserves the possibility to set up the initial states of the integrator chains in the Brunovsky form (9). For observability, recall that a state of (A, B, C) is *unobservable* if the state cannot be uniquely distinguished from the system output.

Let us use $\mathcal{B}$ and $\mathcal{C}$ to denote the range space of the input matrix B and the null space of the output matrix C, respectively. For zero inputs, the state is not uniquely distinguishable from the system output if $\mathcal{C}$ contains a nontrivial A-invariant subspace. This characterization may be extended to the case with unknown inputs, where the indistinguishability arises if $\mathcal{C}$ contains a nontrivial $(A, \mathcal{B})$-invariant subspace. A subspace of states, say $\mathcal{V}$, is A-invariant if $A\mathcal{V} \subset \mathcal{V}$; it is $(A, \mathcal{B})$-invariant if $A\mathcal{V} \subset \mathcal{B} + \mathcal{V}$. A system-theoretic interpretation of an $(A, \mathcal{B})$-invariant subspace contained in $\mathcal{C}$ is that for any given $x_0 \in \mathcal{V}$, there exists an input u_0 such that $Ax_0 + Bu_0$ remains in $\mathcal{V}$ and hence in $\mathcal{C}$. The following result relates A-invariant and $(A, \mathcal{B})$-invariant subspaces to state feedback Basile and Marro (1992):

Lemma 2.9 (state-feedback connection between invariant subspaces) $\mathcal{V}$ *is an* $(A, \mathcal{B})$*-invariant subspace if and only if there exist a (nonunique) state-feedback gain* K *such that* $\mathcal{V}$ *is* $(A - BK)$*-invariant.*

The above lemma shows that the choice of an input u_0 such that $Ax_0 + Bu_0$ remains in $\mathcal{V}$ and hence in $\mathcal{C}$ may be automated by a convenient choice of state feedback.

Using linearity, the sum of two $(A, \mathcal{B})$-invariant subspaces is again an $(A, \mathcal{B})$-invariant subspace. Therefore there must be a unique maximal $(A, \mathcal{B})$-invariant subspace in $\mathcal{C}$, which we shall denote by $\mathcal{V}^*(A, \mathcal{B}, \mathcal{C})$. Since a subspace is maximal if there is no other subspace that strictly includes it, by lemma 2.9 , $\mathcal{V}^*(A, \mathcal{B}, \mathcal{C})$ is also the *maximal unobservable subspace* under state feedback.

Definition 2.10 (strongly observable state-space realization) *An observable state-space realization (A, B, C) is strongly observable if the realization is observable under feedback equivalence transformations.*

Corollary 2.11 (strongly observable state-space realization) *If (A, B, C) is strongly observable, then $\mathcal{V}^*(A, \mathcal{B}, \mathcal{C}) = 0$.*

Lemma 2.12 (strongly observable state-space realization) *Let (A, B, C) be a strongly observable state-space realization of degree n. Then*

$$\operatorname{rank}\begin{bmatrix} \lambda - A & B \\ -C & 0 \end{bmatrix} = n + \operatorname{rank} B \tag{10}$$

for all $\lambda \in \mathbb{C}$.

Proof: (contradiction, continuous time) If λ_0 is a zero frequency, then (10) will lose rank at $\lambda = \lambda_0$, and there will exist a pair of constant vectors x_0 and u_0 such that

$$\begin{bmatrix} \lambda_0 - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

But this means that if we have an input $u(t) = u_0 e^{\lambda_0 t}$, $t \geq 0$, then there exists an initial state x_0 such that the response is $y(t) \equiv 0$, $t > 0$, and hence (A, B, C) cannot be strongly observable.

□

Definition 2.13 (AR and DVR of a system) *Suppose Σ is a system defined as the set of trajectories of its external signals $w \in \mathcal{H}^m(\mathbb{R}, \mathbb{C}m)$.*

1. *Let $P \in \mathbb{C}^{p \times m}[\lambda]$ have full row rank as a polynomial matrix. If*

$$P(\lambda)w(t) = 0 \tag{11}$$

   then (11) is called AR (autoregressive representation) of Σ.

2. *Let A, B, C, and D be given constant matrices. If there exist a pair of variable x and v such that*

$$\begin{aligned} \lambda x(t) &= Ax(t) + Bv(t) \\ w(t) &= Cx(t) + Dv(t), \end{aligned} \tag{12}$$

   then (12) is called DVR (driving variable representation) of Σ. Moreover, if $\dim x$ and $\dim v$ are minimal with respect to every possible DVR of Σ, then (12) is minimal.

Corollary 2.14 (minimal DVR of a system – non-uniqueness) *Any two minimal DVRs of a system may be transformed to one another by feedback equivalence transformations.*

Corollary 2.15 (dimension of the driving variable) *If (12) describes a minimal DVR, then D has full column rank.*

3 Computational issues and the main results

The definitions we used in the frequency-domain part of section 2 rely heavily on unimodular transformations and related canonical forms. These transformations and forms can be constructed by elementary operations (Gantmacher, 1953). Elementary operations on the rows of a polynomial matrix $P(\lambda)$ are:

1. multiplication of a row by a constant $c \neq 0$,
2. addition to any row of any other row multiplied by any arbitrary scalar polynomial $p(\lambda)$,
3. interchange of any two rows.

Elementary operations on columns are defined analogously. From a numerical viewpoint, methods based on computation of elementary operations are not very appealing:

1. Euclidean type algorithms are numerically unstable since pivoting techniques are precluded while some of the pivot coefficients may be very small since the choice of a pivot depends only on its degree,
2. methods using minors are very time consuming since the computing time grows as a factorial of the normal rank of the polynomial matrix.

In the following sections we sketch an algorithmic approach avoiding unimodular transformations and their inherent numerical instability. The idea is to use system-theoretic interpretations to translate problems into state space where efficient numerical methods are available through the linear algebra of constant matrices.

The main result is an algorithm for computing a column-reduced basis of the right null space of a full-row-rank polynomial matrix. The algorithm relies on auxiliary algorithms that do not require the computation of elementary polynomial operations; they rely on invariant subspace methods with orthonormal bases, computationally based on Gram-Schmidt orthonormalization, Householder transformations, and the singular value decomposition. The auxiliary algorithms are elaborated in section 4 and section 5.

Algorithm 3.1 (minimal basis for the right null space of a polynomial matrix) *Let* $P \in \mathbb{C}^{p\times m}[\lambda]$ *have full row rank* p *as a polynomial matrix. Apply:*

1. *algorithm 4.1 to obtain a state-space realization* (A, B, C, D) *in the form of a minimal DVR of* $P(\lambda)w(t) = 0$.
2. *algorithm 5.1 to orthogonally transform* (A, B, C, D) *to a controllability Hessenberg form in order to remove the uncontrollable subsystem, that is,*

$$(A, B, C, D) \longrightarrow (A, B, C, D)_{\mathrm{contr}}$$

3. *algorithm 5.2 to represent the controllable pair* (A, B) *in terms of a right polynomial MFD*

$$Q_{\mathrm{r}}(\lambda)P_{\mathrm{r}}^{-1}(\lambda) = (\lambda I - A)^{-1}B$$

where the polynomial matrices $Q_{\mathrm{r}}(\lambda)$ *and* $P_{\mathrm{r}}(\lambda)$ *are right coprime and column reduced by construction.*

Then a minimal basis for the right null space of $P(\lambda)$ *is described in a polynomial matrix* $R \in \mathbb{C}^{m\times(m-p)}[\lambda]$,

$$R(\lambda) = CQ_{\mathrm{r}}(\lambda) + DP_{\mathrm{r}}(\lambda).$$

That is, $R(\lambda)$ *is column reduced and* $\mathrm{rank} R(\lambda) = m - p$ *for all* $\lambda \in \mathbb{C}$.

4 Successive simplification of driving variable representations in the realization of systems in state space

A polynomial matrix $P \in \mathbb{C}^{p\times m}[\lambda]$ of degree $\deg P = l$ may be defined in terms of a matrix polynomial

$$P(\lambda) \quad = \quad P_l\lambda^l + P_{l-1}\lambda^{l-1} + \cdots + P_1\lambda + P_0$$

and manipulated as an array of coefficient matrices $P_i \in \mathbb{C}^{p\times m}$, $i = 0, 1, \ldots, l$. The following algorithm is based on external equivalence of linear systems in the form (1). The result is a minimal DVR of (1), related to any other minimal DVR of (1) by feedback equivalence transformations.

Algorithm 4.1 (state-space realization) *Let $P \in \mathbb{C}^{p\times m}[\lambda]$ be a full-row-rank polynomial matrix without zero columns.*

1. *Define $A \in \mathbb{C}^{m(l+1)\times m(l+1)}$, $B \in \mathbb{C}^{m(l+1)\times p}$, $C \in \mathbb{C}^{p\times m(l+1)}$, $H \in \mathbb{C}^{m\times m(l+1)}$, $J \in \mathbb{C}^{p\times p}$ such that*

$$\begin{bmatrix} A & B \\ C & 0 \\ H & J \end{bmatrix} := \left[\begin{array}{ccccc|c} 0 & \cdots & \cdots & \cdots & 0 & I \\ I & \ddots & & & \vdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & I & 0 & 0 \\ \hline P_l & \cdots & \cdots & P_1 & P_0 & 0 \\ \hline 0 & \cdots & \cdots & 0 & I & 0 \end{array}\right].$$

2. *Apply algorithm 4.2 to (A,B,C,H,J), the result being (A_{11},B_{12},H_1,J_2). Define new (A,B,C,H,J) such that*

$$\begin{bmatrix} A & B \\ C & 0 \\ H & J \end{bmatrix} \quad := \quad \begin{bmatrix} A_{11}^{\mathrm{T}} & H_1^{\mathrm{T}} \\ B_{12}^{\mathrm{T}} & 0 \\ 0 & -I \end{bmatrix}. \tag{13}$$

3. *Apply algorithm 4.2 to the current (A,B,C,H,J), the result being (A_{11},B_{12},H_1,J_2). Define (A,B,C,D,J) such that*

$$\begin{bmatrix} A & B \\ C & D \\ 0 & J \end{bmatrix} \quad := \quad \begin{bmatrix} A_{11}^{\mathrm{T}} & H_1^{\mathrm{T}} \\ B_{12}^{\mathrm{T}} & J_2^{\mathrm{T}} \\ 0 & I \end{bmatrix}. \tag{14}$$

4. *Apply the orthogonal matrix $T_v = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ —specified by (19) in algorithm 4.2— to obtain*

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ C & 0 & -I \\ 0 & T_1 & T_2 \end{array}\right] := \begin{bmatrix} A_{11}^{\mathrm{T}} & H_1^{\mathrm{T}} \\ B_{12}^{\mathrm{T}} & J_2^{\mathrm{T}} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & T_1 & T_2 \end{bmatrix}.$$

Then $P(\lambda)w(t) = 0$ admits a minimal DVR in the form

$$\begin{aligned} \lambda x(t) &= (A + B_2 C)x(t) + B_1 v(t) \\ w(t) &= T_2 C x(t) + T_1 v(t). \end{aligned} \tag{15}$$

Throughout algorithm 4.1 the dimensions of the externally equivalent realizations are successively deflated using lemma 2.12; a system matrix with full column rank for all $\lambda \in \mathbb{C}$ guarantees existence of a unimodular transformation which validates the deflation (the existence is enough; the actual unimodular matrix is not needed for computation purposes). Actually, the deflation is subject to step 3 and step 4 of algorithm 4.2, the full-column-rank system matrix being defined by $(A_{22}, B_{21}, C_2, 0)$.

Algorithm 4.2 (tool in algorithm 4.1) *Let (A, B, C, H, J) be a DVR with static constraints, the representation being described in the form*

$$\begin{aligned} \lambda x(t) &= Ax(t) + Bv(t) \\ 0 &= Cx(t) \\ w(t) &= Hx(t) + Jv(t). \end{aligned} \tag{16}$$

1. *Find an orthonormal basis $(\xi_1, \ldots, \xi_k)$ for $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C})$.*

2. *Wrap a state-feedback $v \longrightarrow v + Kx$ around (16) such that a state-space realization*

$$\begin{aligned} \lambda x(t) &= (A + BK)x(t) + Bv(t) \\ 0 &= Cx(t) \\ w(t) &= (H + JK)x(t) + Jv(t) \end{aligned} \tag{17}$$

   is obtained with a simple invariant subspace $\mathcal{V}(A + BK) = \mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

3. *Find $(\xi_{k+1}, \ldots, \xi_n)$, the orthonormal complement to the basis of $\mathcal{V}^*(\mathcal{A}, \mathcal{B}, \mathcal{C})$, and apply the orthogonal matrix*

$$T = \left[\ \xi_1 \ldots \xi_k \quad \xi_{k+1} \ldots \xi_n\ \right]$$

   as a similarity transformation matrix in (17). In the new coordinates, (17) appears in the Kalman form

$$\begin{aligned} \lambda \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v(t) \\ 0 &= \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ w(t) &= \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Jv(t) \end{aligned} \tag{18}$$

   where $A_{11} \in \mathbb{C}^{k \times k}$, $0 < k \leq \dim x$.

4. *Orthogonally transform v in (18) such that —in the new coordinates— B_2 is in a column-compressed form: if T_v is the relevant orthogonal transformation matrix, then*

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} T_v \tag{19}$$

   with B_{21} full column rank.

Then (16) is externally equivalent to a not necessarily minimal DVR in the form

$$\begin{aligned} \lambda x(t) &= A_{11}x(t) + B_{12}v(t) \\ w(t) &= H_1 x(t) + J_2 v(t). \end{aligned} \tag{20}$$

In (16)–(20), x and v denote different quantities whose interpretation remains intact throughout the algorithm; $\dim x$ remains constant while $\dim v$ in (18) is reduced to that in (20). In (20), $\dim x$ and $\dim v$ need not be the minimal dimensions amongst all externally equivalent state-space realizations of (16); $\dim x$ is minimal following the second application of algorithm 4.2 in algorithm 4.1.

5 Finalization

The final step in the computation of a minimal basis is a successive backsolve of a set of linear equations in constant matrices. A similarity orthogonal transformation of a state-space realization (A, B, C, D) acts on the full-row-rank blocks in a controllability Hessenberg form to create upper-triangular blocks for the backsolve. The transformations rely on a product of Householder matrices.

Algorithm 5.1 (controllability Hessenberg form) *Let (A, B, C, D) be a state-space realization in the ussual form.*

1. *Change coordinates —using an orthogonal similarity transformation— such that in the new coordinates, (A, B, C, D) is described by*

$$\left[\begin{array}{cccccc|c} A_{11} & \cdots & \cdots & A_{1,\mu} & A_{1,\mu+1} & & B_1 \\ A_{21} & \ddots & & A_{2,\mu} & A_{2,\mu+1} & & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & A_{\mu,\mu-1} & A_{\mu\mu} & A_{\mu,\mu+1} & & 0 \\ 0 & \cdots & 0 & 0 & A_{\mu+1,\mu+1} & & 0 \\ \hline C_1 & \cdots & \cdots & C_\mu & C_{\mu+1} & & D \end{array}\right] \tag{21}$$

with $A_{ij} \in \mathbb{C}^{k_i \times k_j}$ for $i, j = 1, 2, \cdots, \mu$. The integers k_i are defined such that

$$\begin{array}{rcl} k_1 & = & \text{rank } B \\ k_2 & = & \text{rank } A_{i+1,i} \\ & \vdots & \\ k_\mu & = & \text{rank } A_{\mu,\mu-1}. \end{array} \tag{22}$$

The submatrices B_1 and $A_{i+1,i}$, $i = 1, 2, \cdots, \mu - 1$, are in the row echelon form

$$\left[\begin{array}{ccccc} x_{11} \cdots x_{1r_1} & x_{1,r_1+1} \cdots x_{1r_2} & \cdots & x_{1,r_{q-1}+1} \cdots x_{1r_q} \\ & x_{2,r_1+1} \cdots x_{2r_2} & \cdots & x_{2,r_{q-1}+1} \cdots x_{2r_q} \\ & & \cdots & \cdots \\ & & & x_{q,r_{q-1}+1} \cdots x_{qr_q} \end{array}\right] \tag{23}$$

where $q = k_i$, the column indexes satisfy $q \geq r_1 > r_2 > \cdots > r_q \geq 1$, and the entries x_{11}, x_{2,r_1+1}, $\ldots$, $x_{qr_{q+1}}$ are nonzero scalars.

2. *Delete the rows and the columns that intersect at $A_{\mu+1,\mu+1}$ and denote*

$$n = \sum_{i=1}^{k_\mu} k_i$$

3. *Use orthogonal similarity transformation to change the coordinates such that the $A_{i+1,i}$'s are upper triangular, that is, like*

$$A_{i+1,i} = \left[\begin{array}{ccccc} & & \mathbf{x} & x & x \\ & & & \mathbf{x} & x \\ & & & & \mathbf{x} \end{array}\right] \tag{24}$$

where the nonzero elements are marked by **x**.

The result is a controllable state-space realization

$$(A, B, C, D)_{\text{contr}} := \left[\begin{array}{cccc|c} A_{11} & \cdots & \cdots & A_{1,\mu} & B_1 \\ A_{21} & \ddots & & A_{2,\mu} & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & A_{\mu,\mu-1} & A_{\mu\mu} & 0 \\ \hline C_1 & \cdots & \cdots & C_\mu & D \end{array}\right] \tag{25}$$

$A_{ij} \in \mathbb{C}^{k_i \times k_j}$ for $i, j = 1, 2, \ldots, \mu$, the integers k_i satisfy (22), B_1 is in the form (23), and the $A_{i+1,i}$'s are in the form described by (24).

Algorithm 5.2 (left-to-right conversion) *Let (A, B) be a controllable pair in the controllability Hessenberg form (25), featuring the upper-triangular $A_{21}, \ldots, A_{\mu,\mu-1}$ and the row echelon B_1.*

1. *Define $L \in \mathbb{C}^{n \times (m+n-p)}[\lambda]$ such that*

$$\begin{bmatrix} L_0 & L_1(\lambda) & \ldots & L_{\mu-1}(\lambda) & L_\mu(\lambda) \end{bmatrix} = \begin{bmatrix} -B_1 & \lambda I - A_{11} & \cdots & -A_{1,\mu-1} & -A_{1,\mu} \\ 0 & -A_{21} & & -A_{2,\mu-1} & -A_{2,\mu} \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & -A_{\mu,\mu-1} & \lambda I - A_{\mu,\mu} \end{bmatrix}$$

and a partitioned $R \in \mathbb{C}^{(m+n-p)\times(m-p)}[\lambda]$ such that

$$R(\lambda) = \begin{bmatrix} R_0(\lambda) \\ R_1(\lambda) \\ \cdots \\ R_{\mu-1}(\lambda) \\ R_\mu \end{bmatrix} \tag{26}$$

with $R_0(\lambda), \ldots, R_{\mu-1}(\lambda)$ specified by

$$R_i(\lambda) = \begin{bmatrix} 0 & I & 0 \\ X_i(\lambda) & 0 & 0 \end{bmatrix}, \tag{27}$$

$X_i \in \mathbb{C}^{k_{i+1} \times k_{i+1}}[\lambda]$ being unknown polynomial matrices, to be determined in the sequel, and such that

$$R_\mu := \begin{bmatrix} I_{k_\mu} & 0 \end{bmatrix}.$$

2. *Successively backsolve $L(\lambda)R(\lambda) = 0$ for $X_{\mu-1}(\lambda), \ldots, X_1(\lambda)$, and $X_0(\lambda)$ specified by (27). Proceed from $X_{\mu-1}(\lambda)$, obtained as the solution to*

$$\begin{bmatrix} 0 & \ldots & 0 & -A_{\mu,\mu-1} & \lambda I - A_{\mu,\mu} \end{bmatrix} R(\lambda) = 0.$$

3. *Partition $R(\lambda)$ into $P_\mathrm{r} \in \mathbb{C}^{(m-p)\times(m-p)}[\lambda]$ and $Q_\mathrm{r} \in \mathbb{C}^{n \times (m-p)}[\lambda]$,*

$$\begin{bmatrix} P_\mathrm{r} \\ Q_\mathrm{r} \end{bmatrix} := \left[\begin{array}{c} R_0(\lambda) \\ \hline R_1(\lambda) \\ \cdots \\ R_{\mu-1}(\lambda) \\ R_\mu \end{array}\right].$$

Then $Q_\mathrm{r}(\lambda)P_\mathrm{r}^{-1}(\lambda) := (\lambda I - A)^{-1}B$, the polynomial matrices $Q_\mathrm{r}(\lambda)$ and $P_\mathrm{r}(\lambda)$ are right coprime and column reduced by construction.

6 Conclusions

The algorithms avoid computation of elementary polynomial operations by applying an innovative technique for state-space realization, cf. Wolovich (1971). The technique does not require row-reduced forms of polynomial matrices. Minimal externally equivalent realizations are obtained in a successive conversion of state-space realizations in the form of a driving variable representation. The conversion is guided by invariant subspace methods with orthonormal bases. Computationally, the algorithms rely on numerical methods including Gram-Schmidt orthonormalization, Householder transformations, and the singular value decomposition.

The algorithms have been implemented in MATLAB, partly based on the comercial software Grace *et al.* (1990) and the appendix software to Basile and Marro (1992). The implementation is subject to improvements aiming at algorithmization whose numerically backward stability is subject to formal proofs. For comparison to conventional algorithms with unimodular transformations and related canonical forms, the conventional algorithm for state-space realization (Wolovich, 1971) requires generalization to system descriptions in the form (1). In the original, input-output case concerning left MFDs, the algorithm requires transformation to a row reduced polynomial matrix, multiplication by a unimodular matrix, modulo polynomial matrix division, and constant matrix inversion.

The algorithms are useful in a computer aided design of control systems (CADCS) based on polynomial matrices. They allow us to compute singular properties of a general polynomial matrix, along with a minimal (externally) equivalent state-space realization of a system with external variables on equal footing. The realization is in the form of a driving variable representation. The resulting connections are useful as alternative proofs and improve our understanding of polynomial matrices in the context of control systems applications. Without customization, the algorithms may be used for left-to-right and right-to-left conversion to a reduced form of a polynomial MFD. Little customization is needed for a similar application retaining the uncontrollable dynamics of (1).

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