# ADAPTIVE WAVELET SCHEME FOR ELLIPTIC PROBLEMS WITH SINGULAR RIGHT-HAND SIDE

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**Abstract**

**The paper is concerned with the adaptive wavelet schemes for elliptic operator equations. The suitable wavelet bases adapted to homogeneous Dirichlet boundary conditions are constructed and their properties are studied. Numerical examples are presented for problems with a singular right-hand side.**

## 1 Introduction

Wavelets are by now a widely accepted tool in signal processing as well as in numerical simulation. A function $f$ that is smooth, except at some isolated singularities, typically has a sparse representation in a wavelet basis, i.e. only a small number of numerically significant wavelet coefficients carry most of the information on $f$. This compression property of wavelets led to design of adaptive wavelet methods for solving operator equations. In recent years adaptive wavelet methods have been successfully used for solving partial differential as well as integral equations, both linear and nonlinear. It has been shown that these methods converge and that they are asymptotically optimal in the sense that storage and number of floating point operations, needed to resolve the problem with desired accuracy, remain proportional to the problem size when the resolution of the discretization is refined. Thus, the computational complexity for all steps of the algorithm is controlled.

Suitable wavelet bases on bounded domains are needed for these methods. They are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the $n$-dimensional cube. Finally by splitting the domain into not-overlapping subdomains which are images of $(0,1)^n$ under appropriate parametric mappings one can obtain wavelet bases on a fairly general domain. Therefore, the effectiveness of adaptive wavelet methods is strongly influenced by the choice of the interval wavelet basis, in particular by the condition of these bases. In our contribution, we review a construction of spline-wavelet bases on the interval and its adaptation to homogeneous Dirichlet boundary conditions.

The constructed bases are used in adaptive wavelet methods for solving elliptic partial differential equations with singular right-hand sides. The computation is carried out in MATLAB using Wavelet Toolbox.

## 2 Wavelet bases and wavelet transform

This section provides a short introduction to the concept of wavelet bases. Let $V$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_V$ and induced norm $\|\cdot\|_V$. Let $J$ be some index set and let each index $\lambda \in J$ takes the form $\lambda = (j,k)$, where $|\lambda| = j \in \mathbb{Z}$ is *scale* or *level*. Assume that $J$ can be decomposed as $J = \cup_{j \geq j_0} J_j$, where $j_0 \in \mathbb{Z}$ is some coarsest level.

**Definition 1.** Family $\Psi := \{\psi_\lambda \in J\} \subset V$ is called a *wavelet basis* of $V$, if

*i)* $\Psi$ is a *Riesz basis* for $V$, that means $\Psi$ generates $V$, i.e.

$$V = \mathrm{clos}_{\|\cdot\|_V} \mathrm{span}\Psi, \tag{1}$$

and there exist constants $c, C \in (0,\infty)$ such that for all $\mathrm{b} := \{b_\lambda\}_{\lambda \in J} \in l^2(J)$ holds

$$c\,\|\mathrm{b}\|_{l_2(J)} \leq \left\| \sum_{\lambda \in J} b_\lambda \psi_\lambda \right\|_V \leq C\,\|\mathrm{b}\|_{l_2(J)}\,. \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (2)\}$, $C_\psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and $C_\psi/c_\psi$ is called the *condition* of $\Psi$.

*ii)* Basis functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \leq \mathrm{C}2^{-|\lambda|}$ for all $\lambda \in J$, where $\Omega_\lambda$ is support of $\psi_\lambda$.

By the Riesz representation theorem, there exists a unique family of dual functions $\tilde{\Psi} = \left\{\tilde{\psi}_\lambda, \lambda \in \tilde{J}\right\} \subset V$, which are biorthogonal to $\Psi$, i.e. it holds

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_V = \delta_{i,j}\delta_{k,l}, \quad \text{for all} \quad (i,k) \in J, \quad (j,l) \in \tilde{J}. \tag{3}$$

This dual family is also a Riesz basis for $V$ with Riesz bounds $C^{-1}$, $c^{-1}$. The pair $\Psi$, $\tilde{\Psi}$ is often referred to as *biorthogonal system*, $\Psi$ is called *primal* wavelet basis, $\tilde{\Psi}$ is called *dual* wavelet basis. By the above argument, biorthogonality is a necessary for the Riesz basis property (2) to hold. But unfortunately it is not sufficient, see [9].

In many cases, the wavelet system $\Psi$ is constructed with the aid of a multiresolution analysis.

**Definition 2.** A sequence $S = \{S_j\}_{j \in \mathbb{N}_{j_0}}$ of closed linear subspaces $S_j \subset V$ is called a *multiresolution* or *multiscale analysis*, if the subspaces are nested, i.e.,

$$S_{j_0} \subset S_{j_0+1} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots V \tag{4}$$

and $S$ is dense in $V$, i.e.,

$$\operatorname{clos}_V \left( \cup_{j \in \mathbb{N}_{j_0}} S_j \right) = V. \tag{5}$$

The nestedness of the multiresolution analysis implies the existence of the *complement* or *wavelet spaces* $W_j$ such that

$$S_{j+1} = S_j \oplus W_j. \tag{6}$$

We now assume that $S_j$ and $W_j$ are spanned by sets of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in I_j\}, \quad \Psi_j := \{\psi_{j,k}, k \in J_j\}, \tag{7}$$

where $I_j$, $J_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ and the overall wavelet Riesz basis of $V$ is obtained by $\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j$. From the nestedness of $S$ and the Riesz basis property (2), we conclude the existence of bounded linear operators $\mathbf{M}_{j,0} = \left(m_{l,k}^{j,0}\right)_{l \in I_{j+1}, k \in I_j}$ and $\mathbf{M}_{j,1} = \left(m_{l,k}^{j,1}\right)_{l \in I_{j+1}, k \in J_j}$ such that

$$\phi_{j,k} = \sum_{l \in I_{j+1}} m_{l,k}^{j,0} \phi_{j+1,l}, \quad \psi_{j,k} = \sum_{l \in I_{j+1}} m_{l,k}^{j,1} \phi_{j+1,l}. \tag{8}$$

The desired property in applications is the uniform sparseness of $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$, it means that the number of nonzero entries per row and column remains uniformly bounded in $j$. The single-scale and the multiscale bases are interrelated by $\mathbf{T}_{j,s} : l^2(I_{j+s}) \to l^2(I_{j+s})$,

$$\Psi_{j,s} = \mathbf{T}_{j,s} \Phi_{j+s}. \tag{9}$$

$\mathbf{T}_{j,s}$ is called the *multiscale* or the *wavelet transform*.

The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{S}$ with a dual scaling basis $\tilde{\Phi}$ and dual operators $\tilde{\mathbf{M}}_{j,0}$, $\tilde{\mathbf{M}}_{j,1}$.

*Polynomial exactness* of order $N \in \mathbb{N}$ for primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for dual scaling basis is another desired property of wavelet bases in $V \subset L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$. It means that

$$\mathbb{P}_{N-1} \subset S_j, \quad \mathbb{P}_{\tilde{N}-1} \subset \tilde{S}_j, \quad j \geq j_0, \tag{10}$$

where $\mathbb{P}_m$ is the space of all algebraic polynomials on $\Omega$ of degree less or equal to $m$.

# 3 Construction of wavelet bases with boundary conditions

In this section, we briefly review the construction of stable spline-wavelet basis on the interval satisfying homogeneous Dirichlet boundary conditions of the first order from [3, 4]. The primal scaling bases will be the same as bases designed in [1], because they are known to be well-conditioned. Let $N \geq 3$ be the desired order of polynomial exactness of the primal scaling basis and let $\mathbf{t}^j = (t^j_k)_{k=-N+1}^{2^j+N-1}$ be a sequence of knots defined by

$$\begin{aligned} t^j_k &= 0 \quad \text{for} \quad k = -N+2, \dots, 0, \\ t^j_k &= \frac{k}{2^j} \quad \text{for} \quad k = 1, \dots, 2^j - 1, \\ t^j_k &= 1 \quad \text{for} \quad k = 2^j, \dots, 2^j + N - 2. \end{aligned}$$

The corresponding B-splines of order $N$ are defined by

$$B^j_{k,N}(x) := \left(t^j_{k+N} - t^j_k\right) \left[t^j_k, \dots, t^j_{k+N}\right]_t (t-x)^{N-1}_+, \quad x \in [0,1], \tag{11}$$

where $(x)_+ := \max\{0, x\}$ and $[t_1, \dots t_N]_t f$ is the $N$-th divided difference of $f$. The set $\Phi_j$ of primal scaling functions is then simply defined as

$$\phi_{j,k} = 2^{j/2} B^j_{k,N}, \quad \text{for} \quad k = -N+2, \dots, 2^j - 2, \quad j \geq 0. \tag{12}$$

Thus there are $2^j - N + 1$ inner scaling functions and $N-2$ functions on each boundary. The inner functions are translations and dilations of a function $\phi$ which corresponds to the primal scaling function constructed by Cohen, Daubechies, Feauveau in [5]. In the following, we consider $\phi$ from [5] which is shifted so that its support is $[0, N]$. Figure 3 shows the primal scaling basis for $N = 3$ and $j = 3$.

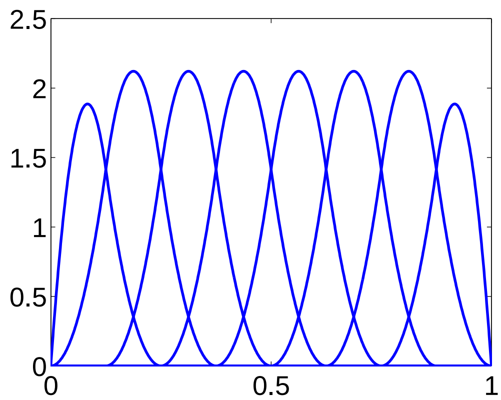


Figure 1: The primal scaling basis for $N = 3$ and $j = 3$

The desired property of the dual scaling basis $\tilde{\Phi}$ is biorthogonality to $\Phi$ and polynomial exactness of order $\tilde{N}$. Let $\tilde{\phi}$ be dual scaling function which was designed in [5] and which is shifted so that its support is $\left[-\tilde{N}+1, N+\tilde{N}-1\right]$. In this case $\tilde{N} \geq N$ and $\tilde{N} + N$ must be an even number. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} = 2^{j/2} \tilde{\phi}\left(2^j \cdot - k\right), \quad k = \tilde{N} - 1, \dots 2^j - N - \tilde{N} + 1. \tag{13}$$

There will be two types of basis functions at each boundary. Basis functions of the first type are defined to preserve polynomial exactness in the same way as in [12]:

$$\theta_{j,k} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle p_{k+N-2}^{\tilde{N}-1}, \phi\left(\cdot - l\right) \right\rangle \tilde{\phi}\left(2^j \cdot - l\right)|_{[0,1]}, \quad k = 2-N, \ldots, \tilde{N} - N + 1. \tag{14}$$

Here $p_0^{\tilde{N}-1}, \ldots, p_{\tilde{N}-1}^{\tilde{N}-1}$ is a basis of $\mathbb{P}_{\tilde{N}-1}\left([0,1]\right)$. As in [12], $p_k^{\tilde{N}-1}$ are Bernstein polynomials defined by

$$p_k^{\tilde{N}-1}\left(x\right) := b^{-\tilde{N}+1} \binom{\tilde{N}-1}{k} x^k \left(b - x\right)^{\tilde{N}-1-k}, \quad k = 0, \ldots, \tilde{N}-1, \tag{15}$$

because they are known to be well-conditioned on $[0,b]$ relative to the supremum norm. In our numerical experiments the choice $b = 10$ seems to be optimal.

The basis functions of the second type are defined as

$$\theta_{j,k} = 2^{\frac{j+1}{2}} \sum_{l=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot -2k - l\right)|_{[0,1]}, \quad k = \tilde{N} - N + 2, \ldots, \tilde{N} - 2, \tag{16}$$

where $\tilde{h}_l$ are scaling coefficients corresponding to $\tilde{\phi}$.

The boundary functions at the right boundary are defined to be symmetrical with the left boundary functions:

$$\theta_{j,k} = \theta_{j,2^j-N+1-k}\left(1 - \cdot\right), \quad k = 2^j - N - \tilde{N} + 2, \ldots, 2^j - 2. \tag{17}$$

Since the set $\Theta_j := \left\{\theta_{j,k} : k = -N+2, \ldots, 2^j - 2\right\}$ is not biorthogonal to $\Phi_j$, we derive a new set $\tilde{\Phi}_j$ from $\Theta_j$ by biorthogonalization. Let $\mathbf{A}_j = \left(\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle\right)_{j,l=-N+2}^{2^j-2}$, then viewing $\tilde{\Phi}_j$ and $\Theta_j$ as column vectors we define

$$\tilde{\Phi}_j := \mathbf{A}_j^{-T} \Theta_j, \tag{18}$$

assuming that $\mathbf{A}_j$ is invertible, which is the case of all choices of $N$, $\tilde{N}$ in our numerical experiments.

Our next goal is to determine the corresponding wavelets. We follow a general principle called stable completion which was proposed in [2]. We found the initial stable completion by the method from [12]. Some of the constructed wavelets are shown in Figure 3. Multivariate wavelet bases on $(0,1)^n$ can be constructed by tensor product.

The condition of scaling and single-scale wavelet bases can be found in [3]. The other criteria for the effectiveness of wavelet bases is the condition number of the corresponding stiffness matrix. Here, let us consider the stiffness matrix for the Poisson equation:

$$\mathbf{A}_{j_0,s} = \left(\left\langle \psi_{j,k}', \psi_{l,m}' \right\rangle\right)_{\psi_{j,k}, \psi_{l,m} \in \Psi_{j_0,s}}, \tag{19}$$

where $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ denotes the multiscale basis. It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use the diagonal matrix for preconditioning

$$\mathbf{A}_{j_0,s}^{prec} = \mathbf{D}_{j_0,s}^{-1} \mathbf{A}_{j_0,s} \mathbf{D}_{j_0,s}^{-1}, \quad \mathbf{D}_{j_0,s} = \operatorname{diag}\left(\left\langle \psi_{j,k}', \psi_{j,k}' \right\rangle^{1/2}\right)_{\psi_{j,k} \in \Psi_{j_0,s}}. \tag{20}$$

Condition numbers of resulting matrices are listed in Tables 1 and 2.

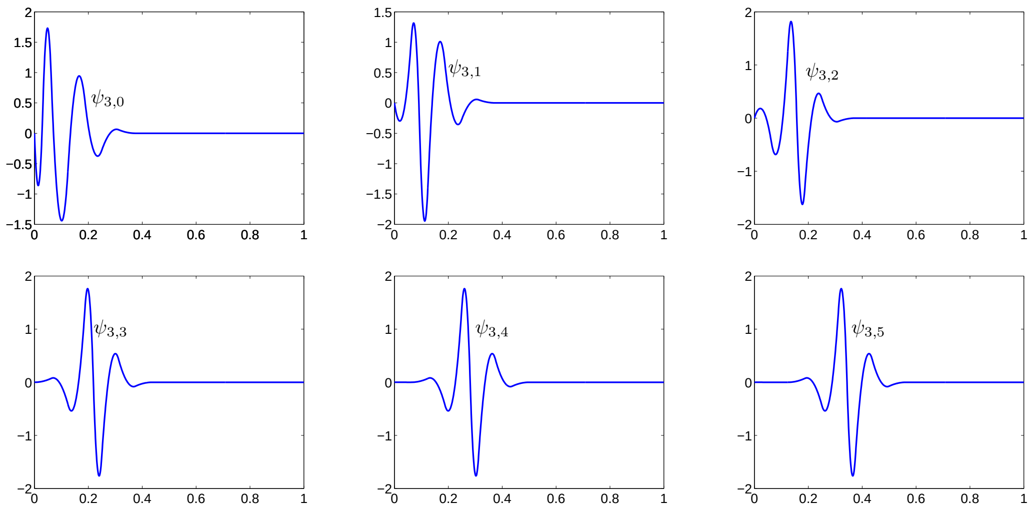


Figure 2: Some quadratic primal wavelets with $\tilde{N} = 5$ vanishing moments satisfying homogeneous Dirichlet boundary conditions of the first order

| $N$ | $\tilde{N}$ | $j$ | $s$ | $M$ | $\mathbf{A}_{j,s}^{prec}$ | $N$ | $\tilde{N}$ | $j$ | $s$ | $M$ | $\mathbf{A}_{j,s}^{prec}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 1 | 16 | 12.24 | 4 | 4 | 4 | 1 | 33 | 47.02 |
| | | | 3 | 64 | 12.72 | | | | 3 | 125 | 49.56 |
| | | | 5 | 256 | 12.85 | | | | 5 | 513 | 50.17 |
| | | | 7 | 1024 | 12.86 | | | | 7 | 2049 | 50.28 |
| 3 | 5 | 4 | 1 | 32 | 52.97 | 4 | 6 | 4 | 1 | 33 | 48.98 |
| | | | 3 | 128 | 54.88 | | | | 3 | 125 | 49.56 |
| | | | 5 | 512 | 55.19 | | | | 5 | 513 | 50.17 |
| | | | 7 | 2048 | 55.24 | | | | 7 | 2049 | 50.28 |
| 3 | 7 | 4 | 1 | 32 | 71.07 | 4 | 8 | 5 | 1 | 65 | 205.56 |
| | | | 3 | 128 | 71.86 | | | | 3 | 257 | 208.37 |
| | | | 5 | 512 | 71.91 | | | | 5 | 1025 | 209.12 |
| | | | 7 | 2048 | 71.91 | | | | 7 | 4097 | 209.31 |

Table 1: The condition number of 1D stiffness matrices $\mathbf{A}_{j,s}^{prec}$ of the size $M \times M$

# 4 Adaptive wavelet scheme

In this section, we briefly review adaptive wavelet methods for the elliptic operator equations similar to the method proposed by Cohen, Dahmen and DeVore in [6, 7, 8]. Our intention is to show the dependence of the effectiveness of these methods on the condition of wavelet bases and to identify the routines which will be used in numerical examples.

Let $H$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$. Let $A : H \to H'$ be the selfadjoint and $H$-elliptic operator, i.e.

$$a(v,w) := \langle Av, w \rangle \lesssim \|v\|_H \|w\|_H \quad \text{and} \quad a(v,v) \sim \|v\|_H^2 . \tag{21}$$

By the Lax-Milgram theorem, $A$ is an isomorphism from $H$ to $H'$, i.e. there exist positive constants $c_A$ and $C_A$ such that

$$c_A \|v\|_H \leq \|Av\|_{H'} \leq C_A \|v\|_H , \quad v \in H. \tag{22}$$

| $N$ | $\tilde{N}$ | $j$ | $s$ | $M$ | $\mathbf{A}_{j,s}^{prec}$ |
|---|---|---|---|---|---|
| 3 | 3 | 3 | 1 | 256 | 62 |
| | | | 2 | 1024 | 94 |
| | | | 3 | 4096 | 132 |
| | | | 4 | 16384 | 172 |
| 3 | 5 | 4 | 1 | 1024 | 99 |
| | | | 2 | 4096 | 295 |
| | | | 3 | 16384 | 557 |
| | | | 4 | 65536 | 929 |
| 4 | 4 | 4 | 1 | 1024 | 933 |
| | | | 2 | 4096 | 1035 |
| | | | 3 | 16384 | 1749 |
| | | | 4 | 65536 | 2792 |

Table 2: The condition number of 2D stiffness matrices $\mathbf{A}_{j,s}^{prec}$ of the size $M \times M$

Therefore, the equation

$$Au = f \tag{23}$$

has for any $f \in H'$ a unique solution. If (22) holds, then (23) is called *well-posed* (on $H$). Typical examples are second order elliptic boundary value problems with homogeneous Dirichlet boundary conditions on some open domain $\Omega \subset \mathbb{R}^d$. In this case $H = H_0^1(\Omega)$ and $H' = H^{-1}(\Omega)$. Other examples are singular integral equations on the boundary $\partial\Omega$ with $H = H^{-1/2}(\partial\Omega)$, $H' = H^{1/2}(\partial\Omega)$.

Thus $H$ is typically a Sobolev space. In the following, we assume that

$$H \subset L^2 \subset H' \quad \text{or} \quad H' \subset L^2 \subset H. \tag{24}$$

We assume that $\mathbf{D}^{-t}\Psi$ is a wavelet basis in the energy space $H$. Thus, we have

$$c_\psi \|\mathbf{v}\|_{l^2} \le \left\|\mathbf{v}^T\mathbf{D}^{-t}\Psi\right\|_H \le C_\psi \|\mathbf{v}\|_{l^2}, \quad \mathbf{v} \in l^2(\mathcal{J}), \tag{25}$$

where $c_\psi > 0$. Then the original equation (23) can be reformulated as an equivalent biinfinite matrix equation

$$\mathbf{Au} = \mathbf{f}, \tag{26}$$

where $\mathbf{A} = \mathbf{D}^{-t}\langle A\Psi, \Psi\rangle \mathbf{D}^{-t}$ is a diagonally preconditioned stiffness matrix, $u = \mathbf{u}^T D^{-t}\Psi$ and $\mathbf{f} = \mathbf{D}^{-t}\langle f, \Psi\rangle$. The following theorem from [11] will be crucial in what follows.

**Theorem 3.** *Under the above assumptions, $u$ solves (23) if and only if $\mathbf{u}$ solves the matrix equation (26). Moreover, the matrix $\mathbf{A}$ satisfies*

$$\|\mathbf{A}\|_{l^2}, \left\|\mathbf{A}^{-1}\right\|_{l^2} \le \frac{C_\psi^2 C_A}{c_\psi^2 c_A} < +\infty. \tag{27}$$

*As an immediate consequence there exists a finite number $\kappa$ such that all finite sections*

$$\mathbf{A}_\Lambda := \mathbf{D}^{-t}\langle A\Psi_\Lambda, \Psi_\Lambda\rangle \mathbf{D}^{-t}, \quad \Psi_\Lambda := \{\psi_\lambda, \lambda \in \Lambda\}, \quad \Lambda \subset \mathcal{J}, \tag{28}$$

*have uniformly bounded condition numbers*

$$\kappa(\mathbf{A}_\Lambda) \le \frac{C_\psi^2 C_A}{c_\psi^2 c_A}, \quad \Lambda \subset \mathcal{J}. \tag{29}$$

Thus the original problem is equivalent to the well-posed problem in $l^2$.

While the classical adaptive methods uses refining and derefining step by step a given mesh according to a-posteriori local error indicators, the wavelet approach is somewhat different and follows a paradigm which comprises the following steps:

1. One starts with a variational formulation but instead of turning to a finite dimensional approximation, using the suitable wavelet basis the continuous problem is transformed into an infinite-dimensional $l^2$-problem, which is well-conditioned.
2. One then tries to devise a *convergent iteration* for the $l^2$-problem.
3. Finally, one derives a practicle version of this idealized iteration. All infinite-dimen-sional quantities have to be replaced by finitely supported ones and the routine for the application of the biinfinite-dimensional matrix $\mathbf{A}$ approximately have to be designed.

The simplest convergent iteration for the $l^2$-problem is a *Richardson iteration* which has the following form:

$$\mathbf{u}_0 := 0, \quad \mathbf{u}_{n+1} := \mathbf{u}_n + \omega \left( \mathbf{f} - \mathbf{A}\mathbf{u}_n \right), \quad n = 0, 1, \ldots. \tag{30}$$

For the convergence, the relaxation parameter $\omega$ has to satisfy

$$\rho := \left\| \mathbf{I} - \omega \mathbf{A} \right\|_{\mathcal{L}(l^2)} < 1. \tag{31}$$

Then the iteration (30) convergence with an error reduction per step

$$\left\| \mathbf{u}_{n+1} - \mathbf{u} \right\|_{l^2} \leq \rho \left\| \mathbf{u}_n - \mathbf{u} \right\|_{l^2}. \tag{32}$$

In the case that $\mathbf{A}$ is symmetric and positive definite, then (31) is satisfied if

$$0 < \omega < \frac{2}{\lambda_{max}}, \tag{33}$$

where $\lambda_{max}$ is the largest eigenvalue of $\mathbf{A}$. It is known that the optimal relaxation parameter is given by

$$\hat{\omega} = \frac{2}{\lambda_{min} + \lambda_{max}}, \tag{34}$$

where $\lambda_{min}$ is the smallest eigenvalue of $\mathbf{A}$. For $\hat{\omega}$ the estimate of the error reduction can be computed as

$$\rho\left(\hat{\omega}\right) = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = \frac{\kappa\left(\mathbf{A}\right) - 1}{\kappa\left(\mathbf{A}\right) + 1} = 1 - \frac{1}{\kappa\left(\mathbf{A}\right) + 1} \leq 1 - \frac{1}{\frac{C_\psi^2 C_A}{c_\psi^2 c_A} + 1}. \tag{35}$$

We use the following implementable version of the ideal iteration (30). It was proved that such an algorithm converge and is asymptotically optimal.

**Algorithm 4. SOLVE $[\mathbf{A}, \mathbf{f}, \epsilon] \to \mathbf{u}_\epsilon$**

*Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.*

*1. Set $j := 0$, $\mathbf{u}_0 := 0$, $\epsilon_0 := \left\| \mathbf{A}^{-1} \right\|_{\mathcal{L}(l^2)} \left\| \mathbf{f} \right\|_{l^2}$.*

*2. While $\epsilon_j > \epsilon$ do*

$j := j + 1,$

$\epsilon_j := 3\rho^K \epsilon_{j-1}/\theta,$

$\mathbf{f}_j := \mathbf{RHS}[\mathbf{f}, \frac{\theta\epsilon_j}{6\omega K}]$,

$\mathbf{z}_0 := \mathbf{u}_{i-1}$,

*For* $l = 1, \ldots, K$ *do*

$\mathbf{z}_l := \mathbf{z}_{l-1} + \omega \left(\mathbf{f}_j - \mathbf{APPLY}[\mathbf{A}, \mathbf{z}_{l-1}, \frac{\theta\epsilon_j}{6\omega K}]\right)$,

*end for,*

$\mathbf{u}_j := \mathbf{COARSE}[\mathbf{z}_K, (1-\theta)\,\epsilon_j]$,

*end while,*

$\mathbf{u}_\epsilon := \mathbf{u}_j$.

For the subroutines **RHS, APPLY**, and **COARSE** we refer to [7].

# 5 Numerical examples

In this section, our intention is to show that the adaptive wavelet method with our bases realizes the optimal convergence rate.

**Example 5.** As a test example we consider the Poisson equation

$$-u'' = f \quad \text{in} \quad \Omega = (0,1)\,, \quad u(0) = u(1) = 0 \tag{36}$$

with the functional $f$ defined by

$$f(v) = 4v\left(\frac{1}{2}\right) - 2\int_0^1 v(x)\,dx. \tag{37}$$

Then the solution $u$ is given by

$$\begin{aligned} u(x) &= x(1-x) + 2x^2 && x \in [0, 0.5) \\ &= x(1-x) + 2(1-x)^2 && x \in [0.5, 1]\,. \end{aligned} \tag{38}$$

Let us define

$$\mathbf{A} = \mathbf{D}^{-1}\left\langle \Psi', \Psi' \right\rangle \mathbf{D}^{-1}, \quad \mathbf{f} = \mathbf{D}^{-1}\left\langle f, \Psi \right\rangle, \quad \mathbf{D} = \operatorname{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k} \right\rangle^{1/2}\right)_{\psi_{j,k}\in\Psi}. \tag{39}$$

Then the variational formulation of (36) is equivalent to

$$\mathbf{AU} = \mathbf{f} \tag{40}$$

and the solution $u$ is given by $u = \mathbf{U}\mathbf{D}^{-1}\Psi$. We solve the infinite dimensional problem (40) by means of the routine **SOLVE**.

The solution $u$ has a limited Sobolev regularity, $u \in H^s(\Omega) \cap H_0^1(\Omega)$ only for $s < 1.5$. Thus the linear methods can only converge with limited order. On the other hand, it can be shown that $u \in B_\tau^{s+1}(L^\tau(\Omega))$ for any positive $s$ and $\tau = (s+0.5)^{-1}$. Therefore, we have

$$\|\mathbf{u} - \mathbf{u}_k\|_{l^2} \le C\,(\#\ \mathrm{supp}\ \mathbf{u}_k)^{-n}\,, \tag{41}$$

where $\mathbf{u}_k$ is the $k$-th approximate iteration. The rate of convergence $n$ is limited only by the polynomial exactness of underlying wavelet bases. It can be shown that in our case relation (41) holds for any $n < N - 1$. Figure 3 shows a logarithmic plot of the realized convergence rate for the spline-wavelet bases designed in this contribution with $N = 3$, $\tilde{N} = 3$ and $N = 4$, $\tilde{N} = 4$.

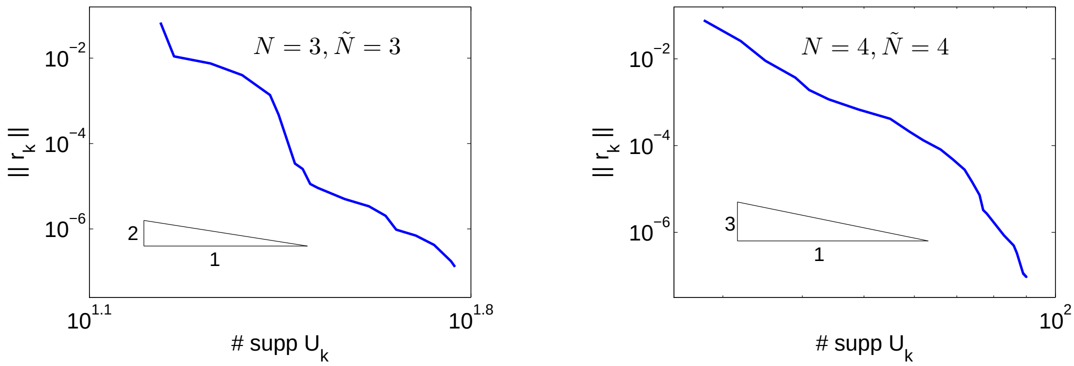


Figure 3: The $l^2$ norm of the residual $\mathbf{r}_k = \mathbf{f} - \mathbf{A}\mathbf{U}_k$ versus the number of degrees of freedom

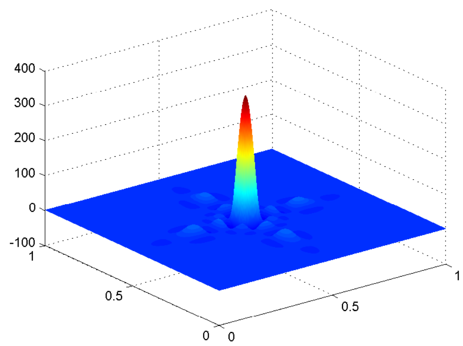


Figure 4: The right-hand side of the equation (42)

**Example 6.** Now, we consider two-dimensional Poisson equation

$$-\Delta u = f, \quad \text{in} \quad \Omega = (0,1)^2, \quad \partial\Omega = 0, \tag{42}$$

with the singular right-hand side, see Figure 4.

We use the above adaptive wavelet scheme and the quadratic wavelet basis with $\tilde{N} = 5$. It can be shown that in this case relation (41) holds for any $n < 1$. A logarithmic plot of the realized convergence rate is shown in Figure 5.

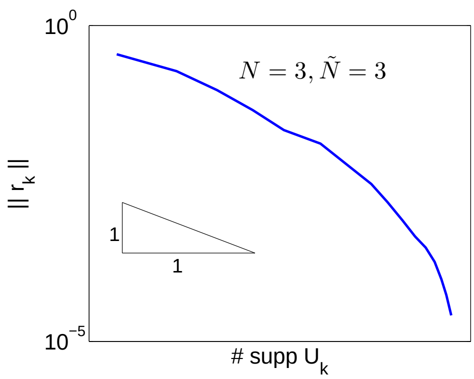


Figure 5: The $l^2$ norm of the residual $\mathbf{r}_k = \mathbf{f} - \mathbf{A}\mathbf{U}_k$ versus the number of degrees of freedom

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