THE SENSITIVITY OF NON-LINEAR ELLIPTIC EQUATION SOLUTIONS ON BOUNDARY CONDITIONS

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Abstract

Modelling of living tissues is the important issue of biomechanics. The non linear continuum mechanics tools are used especially for big deformations description and many nonlinearities in balance equations are present. This contribution is devoted to numerical analysis of simplified elasticity problem where the finite volume and the finite element method approach will be used.

1 Problem Description

Biomechanics modelling of soft living tissues uses non-linear continuum tools. In our problem we tackle the non-linear steady elasticity so let us introduce the balance equation on the reference domain Ω_0 that stays

$$\int_{\Omega_0} (\mathbf{P} : \mathrm{Grad}\, \delta\mathbf{u} - \mathbf{f} \cdot \delta\mathbf{u})\, dV - \int_{\partial\Omega_0} \bar{\mathbf{T}} \cdot \delta\mathbf{u}\, dS = 0\,, \quad \mathbf{u} = \bar{\mathbf{u}} \ \wedge \ \delta\mathbf{u} = 0\,, \text{ on } \partial\Omega_{0\,\mathbf{u}}\,, \tag{1}$$

which we can have also in equivalent form assuming the displacement field is a vector valued function smooth enough respect to spatial reference variables

$$\begin{aligned} \mathrm{Div}\, \mathbf{P} + \mathbf{f} &= 0\,, \text{ in } \Omega_0\,, \\ \mathbf{u} &= \bar{\mathbf{u}}\,, \text{ on } \partial\Omega_{0\,\mathbf{u}}\,, \\ \mathbf{PN} &= \bar{\mathbf{T}}\,, \text{ on } \partial\Omega_{0\,\mathbf{P}}\,. \end{aligned} \tag{2}$$

with $\mathbf{P}$, $\mathbf{f}$ and $\bar{\mathbf{T}}$, the first Piola–Kirchhoff stress, the bulk load and the boundary traction respectively. Here $\mathbf{u}$ denotes the displacement field, $\mathbf{N}$ the unit normal vector to boundary surface $\partial\Omega_0$ and $\delta\mathbf{u}$ are test functions, for further details see [Ger00]. In our problem we use classical compressible isotropic Hookean material. If we simplify the three dimensional case described by (1) and (2) into one dimension we will receive

$$\frac{\Upsilon}{2} \int_a^b (u'(x) + 1)((u'(x) + 1)^2 - 1) v'(x)\, dx = \int_a^b f(x) v(x)\, dx\,, \text{ for all } v(x) \text{ test}\,, \tag{3}$$

on the reference interval $x \in (a, b)$ with Υ the stiffness modulus. As in the three dimensional case we can write for the classical form

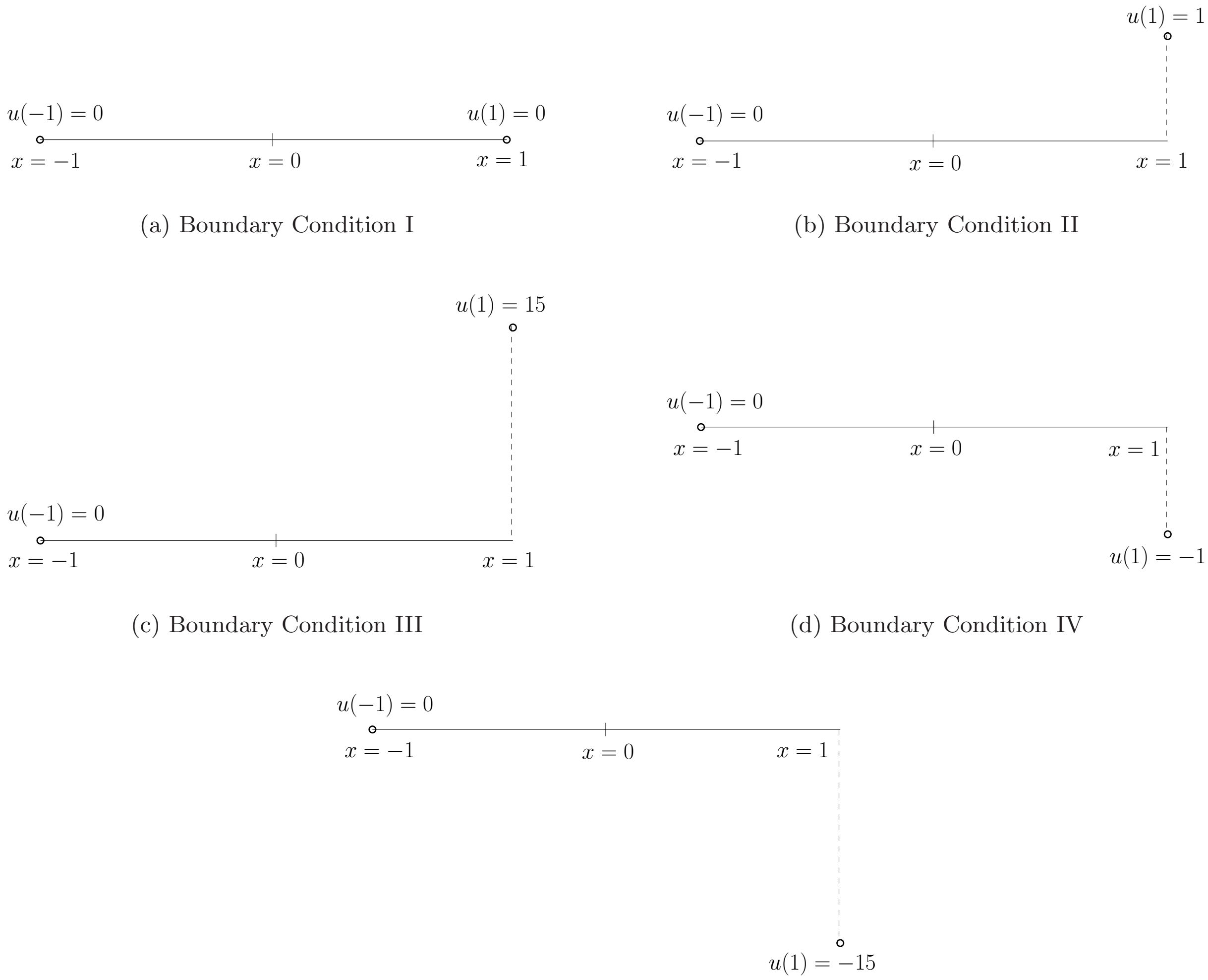
$$-\frac{\Upsilon}{2} \left((u'(x) + 1)((u'(x) + 1)^2 - 1)\right)' = f(x)\,, \quad x \in (a, b) \tag{4}$$

with $f(x)$ the bulk load and Dirichlet boundary conditions $u(a) = u_a, u(b) = u_b$ for (3) (considering $v(a) = v(b) = 0$) and (4). As the other simplification we do not consider any traction on the boundary, so the Neumann boundary conditions are neglected.

2 Numerical Experiments

In our experiments we were faced to many convergence problems. For three basic configurations distinguished by different bulk load f there were performed numerical tests for five boundary condition prescriptions shown in Figure 1.

For solution of (4) we have used *MATLAB* program where were been implemented the *finite volume method* described in [Rob03] and the *discrete duality finite volume method* introduced in [BA07, BH06a]. In *COMSOL* program there were performed experiments on the weak form equation (3) with an advantage of just implemented the *finite element method*. As domain for boundary value problem in strong and weak form we have chosen an open interval $(-1, 1)$.



(a) Boundary Condition I

(b) Boundary Condition II

(c) Boundary Condition III

(d) Boundary Condition IV

(e) Boundary Condition V

Figure 1: Tested boundary condition cases

Finite Volume and Element Schemes

For finding the solution of Eq. (4) with the finite volume method (6) or the discrete duality finite volume method it was been used *Newton scheme* (5) for solution of non-linear equation system $\mathbf{F}(\tilde{\mathbf{u}}) = \mathbf{0}$ based on Eq. (6), for further details see [Alf00].

$$\begin{array}{ll} \text{For} & k = 0, \ldots, \text{until convergence}\,, \\ \text{set} & \tilde{\mathbf{u}}^{(k+1)} = \tilde{\mathbf{u}}^{(k)} + d\tilde{\mathbf{u}}^{(k)}\,, \\ \text{solve} & \mathbf{F}'(\tilde{\mathbf{u}}^{(k)})d\tilde{\mathbf{u}}^{(k)} = -\mathbf{F}(\tilde{\mathbf{u}}^{(k)})\,. \end{array} \tag{5}$$

Here $\tilde{\mathbf{u}} \in \mathbb{R}^n$ is a displacement vector with dimension determined by interval discretization, $\tilde{\mathbf{u}}^{(0)} \in \mathbb{R}^n$ is given initial iteration and $\mathbf{F}'(\tilde{\mathbf{u}}) \in \mathbb{R}^n \times \mathbb{R}^n$ is Jacobi matrix of $\mathbf{F}(\tilde{\mathbf{u}})$. Let us note that we obtain finite volume scheme by integration of Eq. (4) over all control volumes K_i of discrete interval.

$$\int_{K_i} -\frac{\Upsilon}{2}\left((u(x)+1)((u(x)+1)^2-1)\right)' \, dx = \int_{K_i} f(x)\, dx \quad \text{for all } i=1,\,\dots\,,n\,. \tag{6}$$

For further details see [Rob03]. As the equation system solver we have used 'backslash' function implemented in MATLAB.

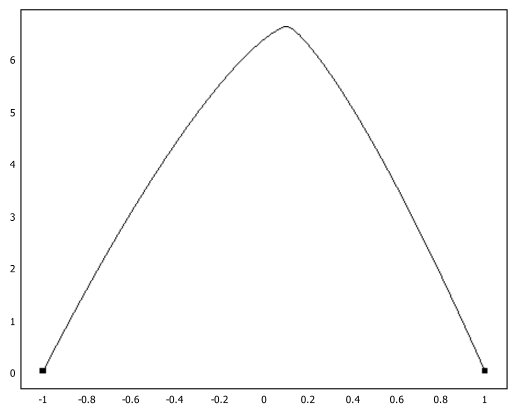
For finite element method we have used COMSOL Multiphysics, where many elements and equation system solvers are just predefined so Eq. (3) was been directly implemented. For further details see almost any finite element method book.

Experiments

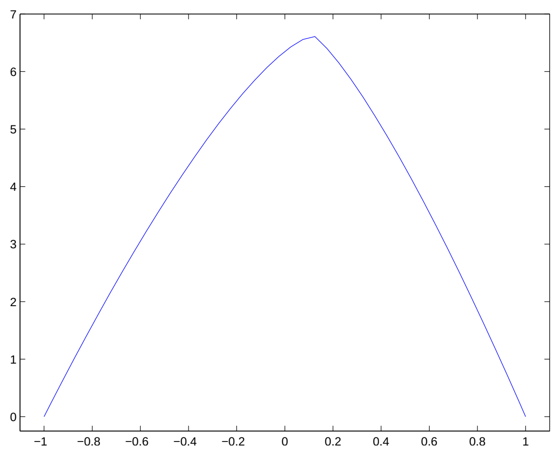
In our experiments for both of the method the stiffness modulus was set to one $\Upsilon = 1$. For the boundary sensitivity confrontation in both used schemes we have taken three different values of the bulk load f. We have used in the case of the finite volume method two discretization steps, 0.1 and 0.05 and in finite element method the Lagrange elements of the second and third degree. The number of iterations for the observed cases are summed up in the Table 1. Every boundary condition from Figure 1 was marked with Roman number from I to V. If the number of Newton iterations was bigger than 25 the problem was considered as non convergent.

FVM – MATLAB										
BC	*I*		*II*		*III*		*IV*		*V*	
step	*0.1*	*0.05*	*0.1*	*0.05*	*0.1*	*0.05*	*0.1*	*0.05*	*0.1*	*0.05*
f=350	14	15	17	16	21	—	14	14	20	27
f=1	19	—	—	22	13	14	—	—	13	15
f=0	1	1	11	12	13	14	11	13	13	15
FEM – COMSOL										
BC	*I*		*II*		*III*		*IV*		*V*	
el.degree	*L2*	*L3*	*L2*	*L3*	*L2*	*L3*	*L2*	*L3*	*L2*	*L3*
f=350	15	—	21	—	—	—	13	—	—	—
f=1	—	—	—	8	5	5	—	—	—	—
f=0	1	1	2	2	2	2	2	2	2	2

Table 1: Number of iterations of FVM implemented in MATLAB and FEM realized by COMSOL

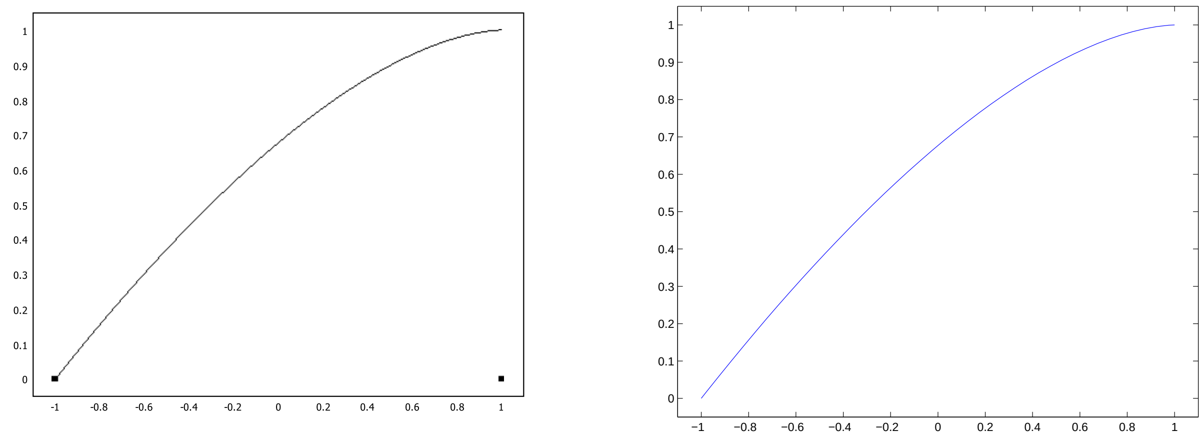


(a) Solution from COMSOL



(b) Solution from MATLAB

Figure 2: Solutions $u(x)$ for BC I for $f = 350$



(a) Solution from COMSOL

(b) Solution from MATLAB

Figure 3: Solutions $u(x)$ for BC II for $f = 1$

3 Conclusion

During our experiments we could see that better finite elements very often do not yield better results and in some cases the method convergent only for the less degree element. Also smaller steps caused many convergence problems for FVM. Except these convergence difficulties the choice of the initial iteration was problematic. We have tried also to implement the discrete duality finite volume method which should be robust, but the numerical results were worst than in choice of finite volume method. If we taken the sensitivity of the numerical solution on BC as the ability of convergence, the most sensible case on boundary changes is for the bulk loads near to $f = 1$ and the best results are for the zero bulk loading $f = 0$, which in mechanics may correspond to boundary tractions only.

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